

# Nonlinear elliptic equations with a singular perturbation on compact Lie groups and homogeneous spaces

Weiping Yan · Yong Li

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**Abstract** This paper is devoted to the study of a class of singular perturbation elliptic type problems on compact Lie groups or homogeneous spaces  $\mathcal{M}$ . By constructing a suitable Nash-Moser-type iteration scheme on compact Lie groups and homogeneous spaces, we overcome the clusters of “small divisor” problem, then the existence of solutions for nonlinear elliptic equations with a singular perturbation is established. Especially, if  $\mathcal{M}$  is the standard torus  $\mathbf{T}^n$  or the spheres  $\mathbf{S}^n$ , our result shows that there is a local uniqueness of spatially periodic solutions for nonlinear elliptic equations with a singular perturbation.

**Keywords** Elliptic equations · singular perturbation · Lie groups · Small divisors

## 1 Introduction and Main Results

The problem of solving nonlinear elliptic equations with a singular perturbation inspired by the work of Rabinowitz[19]. He studied the solvability of the following equation with singular perturbation

$$-\sum_{i,j=1}^n (a_{i,j}(x)u_{x_j})_{x_i} + u = \varepsilon f(x, u, Du, D^2u, D^3u),$$

where  $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ , coefficients  $a_{i,j}$  are periodic in  $x_1, x_2, \dots, x_n$ ,  $\varepsilon \in \mathbf{R}$ , the function  $f$  is also periodic in  $x_1, x_2, \dots, x_n$ . By employing the Nash-Moser iteration process, he proved that above elliptic singular perturbation problem has a uniqueness spatial periodic solution. Han, Hong and Lin[8]

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College of Mathematics, Jilin University, Changchun 130012, P.R. China.  
Beijing International Center for Mathematical Research, Peking University, Beijing 100871, P.R. China.  
E-mail: yan8441@126.com

partially extended the work of Rabinowitz[19], they considered the following singular perturbation problem

$$-\Delta u + u + \varepsilon a(D^p u) = f(x), \quad x \in \mathbf{R}^2,$$

where  $p \geq 4$ , the function  $a(x)$  is smooth and  $f(x)$  is  $(2\pi)^2$ -periodic. Under some assumptions on  $a(x)$  and  $f(x)$ , they employed the Nash-Moser iteration process to prove that above singular problem had spatial periodic solutions. For more related work, we refer to [13,16].

In this paper, we consider the following nonlinear elliptic equations with singular perturbation

$$-\Delta u + u + \varepsilon a(D^{2\varrho} u) = f(x, u), \quad (1)$$

where  $\varrho \in \mathbf{N}$ ,  $x \in \mathcal{M}$ ,  $\mathcal{M}$  is a compact Lie group or, more generally, a compact homogeneous space. The main difficulty is the presence of arbitrarily "small divisors" in the series expansion of the solutions. The operator  $\Delta$  is the Laplace-Beltrami operator defined with respect to a Riemannian metric compatible with the group structure. The nonlinearity is finitely differentiable and vanishes at  $\mathbf{u} = 0$  at least 2. Classical examples of compact connected Lie groups are the standard torus  $\mathbf{T}^n$ , the special orthogonal group  $SO(n)$  and the special unitary group  $SU(n)$ . Examples of compact homogeneous space are the spheres  $\mathbf{S}^n$ , the real and complex Grassmanians, and the moving frames, namely, the manifold of the  $k$ -ples of orthonormal vectors in  $\mathbf{R}^n$  with the natural action of the orthogonal group  $O(n)$ . For more examples, see[5,10].

The information on the spectral analysis of the Laplace-Beltrami operator can be provided by the presence of continuous symmetries expressed via a Lie group action. When the action is transitive (up to isomorphism),

$$\mathcal{M} = (G \times \mathbf{T}^n)/N,$$

where  $G$  is a simply connected compact Lie group,  $\mathbf{T}^n$  is a torus, and  $N$  is a closed subgroup of  $G \times \mathbf{T}^n$ . The functions on  $\mathcal{M}$  can be seen as functions defined on  $G \times \mathbf{T}^n$  and invariant under the action of  $N$ , namely

$$\begin{aligned} \mathbf{L}^2(\mathcal{M}) &= \mathbf{L}^2((G \times \mathbf{T}^n)/N) \\ &= \{u \in \mathbf{L}^2(G \times \mathbf{T}^n) | u(xg) = u(x), \quad \forall x \in G \times \mathbf{T}^n, \quad g \in N\}. \end{aligned} \quad (2)$$

Thus, the Laplace-Beltrami operator on  $\mathcal{M}$  can be identified with the Laplace-Beltrami operator on  $G \times \mathbf{T}^n$ , acting on the functions invariant under  $N$ .

The eigenvalues and the eigenfunctions of the Laplacian on a simply connected compact group  $G$  are, respectively,

$$-|j_1 + \rho|^2 + |\rho|^2, \quad \text{and} \quad \mathbf{e}_{j_1, \sigma}(x_1), \quad x_1 \in G, \quad j_1 \in \Lambda^+(G), \quad \sigma = 1, \dots, d_{j_1},$$

where  $\Lambda^+(G)$  is the cone generated by the natural combinations of the fundamental weights  $w_i \in \mathbf{R}^r$ ,  $i = 1, \dots, r$ .  $r$  denotes the rank of the group, and  $\rho := \sum_{i=1}^r w_i$ . The degeneracy of the eigenvalues is  $d_{j_1} \leq |j_1 + \rho|^{\dim(G)-r}$ . Furthermore, there exists a constant  $D := D(G) \in \mathbf{N}$  such that  $-|j_1 + \rho|^2 + |\rho|^2 \in$

$\mathbf{Z}D^{-1}, \forall j_1 \in \Lambda^+(G)$ . Using the Fubini theorem,  $\mathbf{L}^2(G \times \mathbf{T}^n) = \mathbf{L}^2(G) \times \mathbf{L}^2(\mathbf{T}^n)$ . By (2), we conclude that the eigenvalues and the eigenfunctions of  $-\Delta + 1$  on  $\mathcal{M}$  are, respectively

$$\omega_j^2 := |j_1 + \rho|^2 - |\rho|^2 + |j_2|^2 + 1, \quad \mathbf{e}_{j,\sigma}(x) = \mathbf{e}_{j_1,\sigma}(x_1)e^{ij_2 \cdot x_2}, \quad x = (x_1, x_2) \in G,$$

where the index  $j = (j_1, j_2)$  is restricted to a subset  $\Lambda_{\mathcal{M}} \subset \Lambda^+(G) \times \mathbf{Z}^n$ ,  $j_1 \in \Lambda^+(G) \times \mathbf{T}^n$ ,  $\sigma \subset [1, d_j]$ ,  $d_j := \dim(\mathcal{M}_j)$ ,  $\mathcal{M}_j \subset \mathcal{M}$ ,  $d_j \leq d_{j_1}$ . This property is crucial to Lemma 12 in section 3.

Rescaling in (1) amplitude  $u(x) \mapsto \delta u(x)$ ,  $\delta > 0$ , we solve the following problem

$$-\Delta u + u + \varepsilon a(D^{2\ell}u) = \varepsilon f(\delta, u), \quad (3)$$

where  $a(s) := as^p$ ,  $f(\delta, u) := b(x)s^p + O(\delta)$ ,  $1 \leq p \leq k$  and  $\varepsilon = \delta^{p-1}$ .

In our paper, we will divide into two cases to discuss the existence of solutions for (3). The first case is  $a(x) = ax$ , where  $a \neq 0$  is a constant, then the “small divisor” phenomenon appears. The second case is  $a(\cdot) \in \mathbf{C}^k(\mathbf{R})$ . The second case is simpler than the first case, and we can use the Nash-Moser iteration scheme constructed in the first case to solve it. In what follows, we deal with the first case, i.e.  $a((-1)^\ell \Delta^\ell u) = (-1)^\ell a \Delta^\ell u$ . Thus we can rewrite (3) as

$$-\Delta u + u + (-1)^\ell \varepsilon a \Delta^\ell u = \varepsilon f(\delta, u). \quad (4)$$

Assume that  $a$  is an irrational number and diophantine, i.e. there are constants  $\gamma_0 > 0$ ,  $\tau_0 > 1$ , such that

$$|m - an| \geq \frac{\gamma_0}{|n|^{\tau_0}}, \quad \forall (m, n) \in \mathbf{Z}^2 \setminus \{(0, 0)\}. \quad (5)$$

Then using Lemma 2 (see section 2) there exist  $\gamma > 0$  and  $\tau > 0$  such that the first order Melnikov nonresonance condition

$$|\omega_j^2 + 1 - \varepsilon a \omega_j^{2\ell}| \geq \frac{\gamma}{|j + \vec{\rho}|^\tau}, \quad (6)$$

where  $\omega_j^2 = |j_1 + \rho|^2 - |\rho|^2 + |j_2|^2$ ,  $j = (j_1, j_2) \in \Lambda_{\mathcal{M}} \subset \Lambda^+(G) \times \mathbf{Z}^n$  and  $\vec{\rho} = (\rho, 0)$ .

In this paper, we make more general assumptions on nonlinear terms  $f$ , which include the standard tame estimates and Taylor tame estimates. We assume that the nonlinear terms  $f \in \mathbf{C}^k(\mathcal{M} \times \mathbf{R}, \mathbf{R})$ ,  $f(0, 0) = 0$ ,  $\partial_u f(x, 0) = \dots = (\partial_u^{p-1} f)(x, 0) = 0$ ,  $\partial_u^p f(x, 0) \neq 0$ ,  $1 \leq p \leq k$ ,  $k \geq 2$  and

$$\|\partial_u f(x, u')u\|_s \leq c(s)(\|u\|_s^{p-1} + \|u'\|_s \|u\|_{s_0}^{p-1}), \quad (7)$$

$$\begin{aligned} & \|f(x, u + u') - f(x, u') - D_u f(x, u')u\|_s \\ & \leq c(s)(\|u'\|_s \|u\|_{s_0}^{p-1} + \|u\|_{s_0} \|u\|_s^{p-1}), \end{aligned} \quad (8)$$

where  $s > s_0 > 0$ ,  $p > 1$ ,  $\forall u, u' \in \mathbf{H}_s$  such that  $\|u\|_{s_0} \leq 1$  and  $\|u'\|_{s_0} \leq 1$ . In particular, for  $s_0 = s$ ,

$$\|f(x, u + u') - f(x, u') - D_u f(x, u')u\|_s \leq c(s)\|u\|_s^p.$$

In fact, when  $p = 2$ , assumption (7) and (8) are natural for  $f \in \mathbf{C}^k(\mathcal{M} \times \mathbf{R}, \mathbf{R})$ , which are tame estimates and Taylor tame estimates, respectively.

Our main results are based on the Nash-Moser iterative scheme, which is firstly introduced by Nash[15] and Moser[14]; see[12] for more details. Recently, Berti and Procesi[2] developed suitable linear and nonlinear harmonic analysis on compact Lie groups and homogeneous spaces, and via the technique and the Nash-Moser implicit function theorem, they found a family of time-periodic solutions of nonlinear Schrödinger equations and wave equations. Inspired by the work of [2, 4, 19, 20], we will construct a new suitable Nash-Moser iteration scheme to study the elliptic-type singular perturbation problems (1) on compact Lie groups and homogeneous spaces. Meanwhile, Theorems 3-4 allow more general  $\mathbf{C}^k$  nonlinearities on a higher dimensional space than the work of [8, 19]. Since the proof process of Theorem 3-4 is similar with Theorem 1-2, we omit them. For a general case, we assume that the nonlinear terms satisfy (7)-(8).

To carry out the Nash-Moser iteration scheme, we also need to introduce the Banach scale of the Sobolev spaces on a group. Let  $\mathcal{M} = (G \times \mathbf{T}^n)/N$  be a homogeneous space, where  $G$  is a simply connected Lie group of dimension  $d$  and rank  $r$ . By Theorem 5 in section 2 (Peter-Weyl theorem), we have the orthogonal decomposition

$$\mathbf{L}^2 := \mathbf{L}^2(\mathcal{M}, \mathbf{C}) = \bigoplus_{j \in \Lambda_{\mathcal{M}}} \mathcal{N}_j.$$

The Fourier series of  $u \in \mathbf{L}^2$  is defined by

$$u = \sum_{j \in \Lambda_{\mathcal{M}}} u_j,$$

where  $u_j := \Pi_{\mathcal{N}_j} u$  and  $\Pi_{\mathcal{N}_j} : \mathbf{L}^2 \rightarrow \mathcal{N}_j$  are the spectral projectors,  $\Lambda_{\mathcal{M}} := \{j \in \Lambda^+ \times \mathbf{Z}^n \text{ such that } \mathcal{N}_j \neq \{0\}\}$  is closed under sum.

More precisely, for  $\forall 1 \leq d'_j \leq d_j$ , we have

$$u(x) = \sum_{j \in \Lambda_{\mathcal{M}}} \sum_{\sigma=1}^{d'_j} u_{j,\sigma} \mathbf{e}_{j,\sigma}$$

with the norm

$$\|u(x)\|_{\mathbf{L}^2}^2 = 2\pi \sum_{j \in \Lambda_{\mathcal{M}}} \sum_{\sigma=1}^{d'_j} |u_{j,\sigma}|^2.$$

We define the Sobolev scale of Hilbert spaces

$$\mathbf{H}_s := \mathbf{H}_s(\mathcal{M}, \mathbf{C}) = \{u = \sum_{j \in \Lambda_{\mathcal{M}}} u_j \|u\|_s^2 := \sum_{j \in \Lambda_{\mathcal{M}}} e^{2|j + \vec{\rho}|s} \|u\|_{\mathbf{L}^2}^2 < +\infty\},$$

where  $\vec{\rho} = (\rho, 0) \in \Lambda^+ \times \mathbf{Z}^n$ . It is obvious that  $\mathbf{H}^0 = \mathbf{L}^2$ . Since  $\mathcal{M}$  is a compact  $C^\infty$ -Riemannian manifold without boundary, for any  $s \in \mathbf{N}$ ,  $\mathbf{H}_s$  is equivalent to the usual Sobolev sapce

$$H_s = \{u \in \mathbf{L}^2 | D^\alpha u \in \mathbf{L}^2, \forall |\alpha| \leq s, \|u\|_s^2 := \sum_{|\alpha| \leq s} \|D^\alpha u\|_{\mathbf{L}^2}^2\}.$$

For the case  $a(x) = ax$  in (1), we have the following result.

**Theorem 1** *Let  $\tau, \kappa_0, \delta > 0$  and  $0 < \sigma_0(\mathcal{M}) < \bar{\sigma}(\mathcal{M}) < \sigma(\mathcal{M}) < k(\mathcal{M}) - 1$ . Assume that  $a > 0$  is diophantine. For  $\delta_0 > 0$ ,  $s_0 := \bar{\sigma}(\mathcal{M})$ ,  $k := k(\mathcal{M}) \in \mathbf{N}$  and  $f \in \mathbf{C}^k$  satisfying (7)-(8), Then there exists a positive measure Cantor set  $\mathcal{C} \subset [0, \delta_0]$  such that,  $\forall a \in \mathcal{C}$ ,  $u_\delta(x, \varepsilon)$  is a local uniqueness solution of (4). Furthermore, there exists a curve*

$$u \in \mathbf{C}^1([0, \delta_0]; \mathbf{H}_{s_0}) \text{ with } \|u(\delta)\|_{s_0} = O(\delta).$$

For the second case, we consider equation (3) and assume that  $a \in \mathbf{C}^k(\mathbf{R})$ ,  $a(0) = 0$ , and

$$\|\partial_u a(u')u\|_s \leq c(s)(\|u\|_s^{p-1} + \|u'\|_s \|u\|_{s_0}^{p-1}), \quad (9)$$

$$\|a(u + u') - a(u') - D_u a(u')u\|_s \leq c(s)(\|u'\|_s \|u\|_{s_0}^{p-1} + \|u\|_{s_0} \|u\|_s^{p-1}), \quad (10)$$

where  $s > s_0 > 0$ ,  $1 < p \leq k$ ,  $\forall u, u' \in \mathbf{H}_s$  such that  $\|u\|_{s_0} \leq 1$  and  $\|u'\|_{s_0} \leq 1$ . In particular, for  $s_0 = s$ ,

$$\|a(u + u') - a(u) - D_u a(u)u\|_s \leq c(s)\|u\|_s^p.$$

Then we have

**Theorem 2** *Let  $\tau, \kappa_0 > 0$  and  $0 < \sigma_0(\mathcal{M}) < \bar{\sigma}(\mathcal{M}) < \sigma(\mathcal{M}) < k(\mathcal{M}) - 1$ . There exist  $s_0 := \bar{\sigma}(\mathcal{M})$  and  $k := k(\mathcal{M}) \in \mathbf{N}$  such that for  $f, a \in \mathbf{C}^k$  satisfying (7)-(10), equation (3) has a solution  $u(x) \in \mathbf{H}_{s_0}$ .*

The proof of Theorem 2 is similar to the proof of Theorem 1, hence we omit it.

Especially, if  $\mathcal{M}$  is the standard torus  $\mathbf{T}^n$  or the spheres  $\mathbf{S}^n$ , we obtain the existence of spatially periodic solutions for elliptic equation (1). We also need to divide into two cases to discuss. For the first case, we have

**Theorem 3** *Let  $\tau, \kappa_0 > 0$  and  $0 < \sigma_0 < \bar{\sigma} < \sigma < k - 1$ . Assume that  $a > 0$  is diophantine. For  $\delta_0 > 0$ ,  $s_0 := \bar{\sigma}(\mathcal{M})$ ,  $k := k(\mathcal{M}) \in \mathbf{N}$  and  $f \in \mathbf{C}^k$  satisfying (7)-(8), Then there exists a positive measure Cantor set  $\mathcal{C} \subset [0, \delta_0]$  such that,  $\forall a \in \mathcal{C}$ ,  $u = u_\delta(x, \varepsilon)$  is a unique spatially periodic solution of (1). Furthermore, there exists a curve*

$$u \in \mathbf{C}^1([0, \delta_0]; \mathbf{H}_{s_0}) \text{ with } \|u(\delta)\|_{s_0} = O(\delta).$$

For the second case, we have

**Theorem 4** *Let  $\tau, \kappa_0 > 0$  and  $0 < \sigma_0 < \bar{\sigma} < \sigma < k - 1$ . There exist  $s_0 := \bar{\sigma}$  and  $k \in \mathbf{N}$  such that  $\forall f, a \in \mathbf{C}^k$  satisfying (7)-(10). Then equation (1) has a spatially periodic solution  $u(x) \in \mathbf{H}_{s_0}$ .*

The structure of the paper is as follows: In next section, we present some notations related to Lie group, homogeneous spaces and corresponding Laplace-Beltrami operator properties. Section 3 is devoted to the proof of Theorem 1, where we construct a suitable Nash-Moser iteration scheme. In the last section, we will prove a main Lemma (Lemma 12), which deals with the estimate of the linearized operators and plays a crucial role in the Nash-Moser iteration. The measure estimates is given in the appendix.

## 2 Preliminaries

In this section, we recall some basic conceptions and results in the representation theory of Lie group and homogeneous space, which can be found in the books [5, 7, 17] and the paper [2]. Let  $G$  be a compact topological group, and let  $\mathbf{L}^2(G) := \mathbf{L}^2(G, \mathbf{C})$  be the Lebesgue space defined with respect to the normalized Haar measure  $\mu$  of  $G$ .  $(V, \rho_V)$  denotes a finite-dimensional unitary representation of  $G$ . It is a continuous homomorphism  $x \mapsto \rho_V(x)$  which maps  $G$  into the group of unitary transformations  $U(V) \subset \text{End}(V)$ ; here  $V$  denotes a finite-dimensional complex vector space. For fixed  $\{v_1, \dots, v_n\}$  (orthonormal basis) of  $V$ , we can describe the presentation by the unitary matrices

$$U(x) := U^V(x) := \{U_{l,k}^V(x)\} = \{(\rho_V(x)v_l, v_k)\}, \quad l, k = 1, \dots, n := \dim(V) \quad (11)$$

The following Peter-Weyl Theorem gives the Fourier analysis on the group. In the case of the standard torus, the irreducible representations of a group play the role of the exponential basis.

**Theorem 5** *Let  $\hat{G}$  be the set of equivalence classes of irreducible unitary representations of the compact group  $G$ , for each  $j \in \hat{G}$ , let  $\mathcal{M}_j := \mathcal{M}_{V_j}$ . Then the Hilbert decomposition holds*

$$\mathbf{L}^2(G) = \widehat{\bigoplus_{j \in \hat{G}} \mathcal{M}_j}.$$

For  $f \in \mathbf{L}^2(G)$ , we have the  $\mathbf{L}^2$  convergent “Fourier series”

$$f(x) = \sum_{j \in \hat{G}} \text{tr}(f_j \mathbf{e}_j(x)), \quad f_j := \int_G df(x) \bar{\mathbf{e}}_j(x) \mu,$$

where  $\mathbf{e}_j(x) := (\dim V_j)^{\frac{1}{2}} U_j(x)$  and the matrices  $U_j(x) := U^{V_j}(x)$  are defined in (11). Here the matrix  $\bar{\mathbf{e}}_j(x)$  is the complex conjugate of  $\mathbf{e}_j(x)$ , and  $f_j$  are the Fourier coefficient of  $f(x)$ .

By the Schur orthogonality relations

$$\int_G \text{tr}(A\rho_V(x))\overline{\text{tr}(B\rho_V(x))}d\mu(x) = \frac{\text{tr}(AB^\dagger)}{\dim(V)}, \quad \forall A, B \in \text{End}(V),$$

we have that  $\mathbf{e}_{j,\sigma}(x)$  (the matrix coefficients of  $\mathbf{e}_j(x)$ ),  $\sigma = 1, \dots, \dim V_j^2$  form an  $L^2$ -orthonormal basis for  $\mathcal{M}_j$ .

Next we introduce some properties of Laplace-Beltrami operator on the compact Lie groups  $\mathcal{G} = (G \times \mathbf{T}^n)/N$ , where  $G$  is simply connected and  $N$  is finite and central. Let  $G$  be a simply connected compact Lie group of simple type. Define a Riemannian metric on  $G$  by

$$-(X, Y) := \text{tr}(Ad(X) \circ Ad(Y)),$$

which is negative definite of the killing form, where  $Ad(X)(\cdot) := [X, \cdot]$ . Thus we can define the Laplace-Beltrami operator  $\Delta$  on  $G$  with respect to this metric. The following two results are taken from the book [17].

**Theorem 6** *For a simply connected compact Lie group  $G$  of rank  $r$ , there is a one-to-one correspondence between the set of equivalence classes  $\hat{G}$  of irreducible unitary representations and a discrete cone*

$$\Lambda^+ := \Lambda^+(G) = \{j = \sum_{i=1}^r n_i w_i, n_i \in \mathbf{N}\} \subset \mathbf{R}^r$$

generated by  $r$  independent vectors  $w_i \in \mathbf{R}^r$ .

Above result describes all the irreducible representations. Here the rank of a Lie group  $G$  is defined by the dimension of any maximal connected commutative subgroup of  $G$  (maximal torus). The  $\{w_1, \dots, w_r\}$  are called the fundamental weights of the group, and  $\Lambda^+$  is called the cone of dominant weight. The irreducible representation of  $G$  corresponding to the dominant weight  $j = 0$  is the trivial representations on  $V_0 = \mathbf{C}$ .

The matrix coefficients of an irreducible representation are eigenfunctions of the Laplacian. Due to the Laplacian is a real operator,  $\bar{\mathbf{e}}_{j,\sigma}(x)$  is an eigenvector of  $\Delta$  if  $\mathbf{e}_{j,\sigma}(x) \in \mathcal{M}_j$  is an eigenvector with the same eigenvalue. Thus  $\bar{\mathbf{e}}_{j,\sigma}(x) \in \mathcal{M}_{j'}$  for some  $j' \in \Lambda^+$ . Moreover, since the matrix  $\bar{\mathbf{e}}_{j,\sigma}(x)$  is the dual representation on  $V_j^*$  of the matrix  $\mathbf{e}_j(x)$ , we have  $V_{j'} = V_j^*$ .

**Theorem 7** *Each  $\mathcal{M}_j$  is an eigenspace of the Laplace Beltrami operator  $\Delta$  with eigenvalue*

$$-|j + \rho|^2 + |\rho|^2, \quad \rho := \sum_{i=1}^r w_i,$$

where  $d_j := \dim(\mathcal{M}_j) \leq |j + \rho|^{\dim(G)-r}$ .

Set the positive simple roots  $\alpha_1, \dots, \alpha_r \in \mathbf{R}^r$  of the group  $G$ , then the eigenvalues and the eigenfunctions of the Laplace operator can be described, see[17] for more details. They satisfy the relations

$$(w_i, \alpha_j) = \frac{1}{2} \delta_{i,j} |\alpha_j|^2, \quad \forall i, j = 1, \dots, r,$$

where  $\delta_{i,j}$  denotes the Kronecker symbol. Define the cone

$$\mathcal{R}^+ := \left\{ \alpha = \sum_{i=1}^r n_i \alpha_i, n_i \in \mathbf{N} \right\} \subset \mathbf{R}^r$$

generated by the natural combinations of the positive simple roots. We define the lattice

$$\Lambda := \left\{ j = \sum_{i=1}^r n_i w_i, n_i \in \mathbf{Z} \right\} \subset \mathbf{R}^r$$

generated by the fundamental weights  $w_1, \dots, w_r$  and  $\Lambda^{++} := \rho + \Lambda^+$ .

**Definition 1.** For  $i, j \in \Lambda$ , we say that  $i \geq j$  if  $i = j + \alpha$  for some  $\alpha \in \mathcal{R}^+$ .

The following two results are taking from[2], and we omit the proof.

**Lemma 1** *There is  $c \in (0, 1)$  such that  $\forall i \geq j, i \in \Lambda^+, j \in \Lambda$  with  $(\rho, j) > 0$ , one has  $|i + \rho| \geq c|j + \rho|$ .*

**Lemma 2** *For any simply connected Lie group  $\mathcal{G}$ , there is  $D \in \mathbf{N}$  such that  $(w_i, w_j) \in D^{-1} \mathbf{Z}$ . Hence*

$$|j|^2, |j + \rho|^2, (\rho, j) \in D^{-1} \mathbf{Z}, \quad \forall j \in \Lambda^+.$$

The eigenspaces of the Laplace operator on  $G \times \mathbf{T}^n$  are

$$\mathcal{M}_{j_1} e^{ij_2 \cdot x_2} \text{ with } (x_1, x_2) \in G \times \mathbf{T}^n, (j_1, j_2) \in \Lambda^+ \times \mathbf{Z}^n,$$

the eigenfunctions  $\mathbf{e}_{j_1, \sigma}(x_1) e^{ij_2 \cdot x_2}$ ,  $1 \leq \sigma \leq d_j$ , and the eigenvalues  $-|j_1 + \rho|^2 + |\rho|^2 - |j_2|^2$ .

**Theorem 8** *Let  $H$  be a closed subgroup of a Lie group  $\mathcal{G}$ . Then there is a unique manifold structure on the quotient space  $\mathcal{G}/H$ , such that the projection map  $\pi : \mathcal{G} \rightarrow \mathcal{G}/H$  is a smooth submersion. Moreover, given a biinvariant metric on  $\mathcal{G}$ , the projection  $\pi$  induces on  $\mathcal{G}/H$  a Riemannian structure such that the Laplace-Beltrami operator on  $C^\infty(\mathcal{G}/H, \mathbf{C})$  is identified with the Laplace-Beltrami operator on*

$$C_{inv}^\infty(\mathcal{G}, \mathbf{C}) := \{f \in C^\infty(\mathcal{G}, \mathbf{C}) \text{ such that } f(x) = f(xg), \forall x \in \mathcal{G}, g \in H\},$$

and the diagram commutes:

$$\begin{array}{ccc} C^\infty(\mathcal{G}/H, \mathbf{C}) & \xrightarrow{\pi^*} & C_{inv}^\infty(\mathcal{G}, \mathbf{C}) \\ \Delta_{\mathcal{G}/H} \downarrow & & \Delta_G \downarrow \\ C^\infty(\mathcal{G}/H, \mathbf{C}) & \xrightarrow{\pi^*} & C_{inv}^\infty(\mathcal{G}, \mathbf{C}). \end{array}$$



The action  $(g, x) \mapsto gx$  of a group  $\mathcal{G}$  on a set  $X$  is called transitive if,  $\forall x \in X$ , the orbit  $\mathcal{O}(x) := \{gx \in X, g \in \mathcal{G}\} = X$ .

**Definition 2.** A compact manifold  $\mathcal{M}$  is said to be homogeneous if there is a compact Lie group  $G$  which acts on  $\mathcal{M}$  transitively and differentiably; that is, for each  $g \in G$ , the map  $x \mapsto gx$  is differentiable in  $\mathcal{M}$ .

The action of  $G \times \mathbf{T}^n$  on any  $p \in \mathcal{M}$  induces a diffeomorphism  $\mathcal{M} \leftrightarrow (G \times \mathbf{T}^n)/N$ , where  $N := N_1 \mathcal{G}_p$ ,  $N_1$  is the finite central subgroup,  $\mathcal{G}_p$  is the stabilizer of  $p$ , the group  $\mathcal{G} = (G \times \mathbf{T}^n)/N_1$ . Theorem 8 shows that a biinvariant metric on  $G \times \mathbf{T}^n$  induces a metric on  $(G \times \mathbf{T}^n)/N$  and, then, on  $\mathcal{M}$  (see [3]).

By Theorem 5 (Peter-Weyl theorem), we have the spectral theory of the Laplace-Beltrami operator on a compact homogeneous space.

**Theorem 9** *The following sum decomposition holds:*

$$L^2(\mathcal{M}) = \widehat{\bigoplus_{j \in \Lambda_{\mathcal{M}}} \mathcal{N}_j}.$$

A basis for  $\mathcal{N}_j \subset \mathcal{M}_j$  is, up to a reordering of the index  $\sigma$ ,

$$e_{j,\sigma}(x) = e_{j_1,\sigma}(x_1) e^{ij_2 \cdot x_2}, \quad \sigma = 1, \dots, d'_j,$$

for some  $1 \leq d'_j \leq d_j$ , where the subspace of functions  $\mathcal{N}_j \subset \mathcal{M}_j := \mathcal{M}_{j_1} e^{ij_2 \cdot x_2}$  defined by

$$\mathcal{N}_j := \text{Span}\{(\rho_{v_j}(x)w_k, v_l), k = 1, \dots, \dim(W), l = 1, \dots, \dim(V)\},$$

the subspace  $W_j := \{w \in V_j | \rho_{V_j}(g)w = w, \forall g \in H\} \subset V_j$ ,  $(v_l)_{l=1, \dots, \dim(V_j)}$  and  $(w_l)_{l=1, \dots, \dim(W_j)}$  are a basis of  $V_j$  and  $W_j$ , respectively. Moreover, each  $\mathcal{N}_j$  is an eigenspace of the Laplacian with dimension  $\dim \mathcal{N}_j \leq \dim \mathcal{M}_j$  and  $\dim \mathcal{N}_j = \dim V_j \dim W_j$ .

Let  $a = \text{tr}(A \rho_{V_i}) \in \mathcal{M}_i$  and  $b = \text{tr}(B \rho_{V_j}) \in \mathcal{M}_j$  denote two eigenfunctions of the Laplace-Beltrami operator  $\Delta$ . The product is given by

$$ab = \text{tr}(A \otimes B \rho_{V_i \otimes V_j}).$$

Then  $V_i \otimes V_j$  can be expressed as the direct sum of irreducible representations

$$V_i \otimes V_j = \bigoplus_{l \in \Lambda^+} V_l^{c_{i,j}^l}, \quad c_{i,j}^l \in \mathbf{Z} \cup \{0\}, \quad (12)$$

where  $c_{i,j}^l$  are called the Clebsch-Gordan coefficients of the group.

Using a theorem of Cartan (see [17], p.345), one can verify that the product of two eigenfunctions is a finite sum of eigenfunctions.

**Lemma 3** *Let  $a \in \mathcal{M}_i$  and  $b \in \mathcal{M}_j$ . Then  $ab \in \bigoplus_{l \leq i+j} \mathcal{M}_l$ .*

The  $\mathbf{L}^2$ -orthogonal projection of  $ab$  on the eigenspace  $\mathcal{M}_l$  is

$$\prod_{\mathcal{M}_l} ab = \sum_{s \leq c_{ij}^l} \text{tr}((A \otimes B)|_{l,s} \rho_{V_l}),$$

where  $(A \otimes B)|_{l,s}$  denotes the restriction of  $A \otimes B$  to the  $s$ th copy of  $V_l$  in (12).

Specially, if  $A = B = Id$ , we obtain the formula for the characters  $\chi_i := \chi_{V_i}$ , namely,

$$\chi_i \chi_j = \sum_{l \leq i+j} c_{ij}^l \chi_l.$$

**Lemma 4** For  $s \geq s_0 > \frac{\dim(\mathcal{M})}{2}$ ,  $\forall u_1, u_2 \in \mathbf{H}_s$ , there hold:

$$\begin{aligned} \|u_i\|_{L^\infty} &\leq c(s) \|u_i\|_s, \quad i = 1, 2, \\ \|u_1 u_2\|_s &\leq c(s) \|u_1\|_s \|u_2\|_s, \\ \|u_1 u_2\|_s &\leq c(s, s_0) (\|u_1\|_s \|u_2\|_{s_0} + \|u_1\|_{s_0} \|u_2\|_s). \end{aligned}$$

**Lemma 5** We define  $J := \Lambda \times \mathbf{Z}^n$  and  $J^+ := \Lambda^+ \times \mathbf{Z}^n$ , for given  $a \in \mathcal{N}_j$  and  $b \in \mathcal{N}_{j'}$ . Then

$$ab \in \bigoplus_{\tilde{j} \in D(j, j')} \mathcal{N}_{\tilde{j}},$$

where for  $j = (j_1, j_2)$ ,  $j' = (j'_1, j'_2)$ ,  $\tilde{j} = (\tilde{j}_1, \tilde{j}_2)$ ,

$$D(j, j') := \{\tilde{j} \in J^+ | \tilde{j}_1 \leq j_1 + j'_1, \tilde{j}_2 = j_2 + j'_2\},$$

Here for given  $j = (j_1, j_2)$ ,  $j' = (j'_1, j'_2) \in J$ , we say that  $j \geq j'$  if  $j_1 \geq j'_1$  and  $|j_2| \geq |j'_2|$ .

The following result shows that the embedding  $\mathbf{H}_s \hookrightarrow \mathbf{C}(\mathcal{M})$  for  $2s > \dim(G \times \mathbf{T}^n) > \dim(\mathcal{M})$ . We denote  $J^+ := \Lambda_{\mathcal{M}}$ . By a small modification of the proof of Lemma 2.15 in [2], we have

**Lemma 6** Let  $2s > d + n + 1$ . For  $u \in \bigoplus_{j \geq j_0, j \in J^+} \mathcal{N}_j$  with  $j_0 = (j_{01}, j_{02}) \in \Lambda^+(G) \times \mathbf{Z}^n$ , and  $(\rho, j_{01}) \geq 0$ , we have

$$\|u\|_{L^\infty} \leq c(s) \|u\|_s e^{-(s - \frac{d+n+1}{2})|j_0|}.$$

Due to the orthogonal splitting

$$\mathbf{H}_s = \bigoplus_{j \in J^+} \mathcal{N}_j,$$

we identify a linear operator  $A$  acting on  $\mathbf{H}_s$  with its matrix representation  $A = (A_j^{j'})_{j, j' \in J^+}$  with blocks  $A_j^{j'} \in \mathcal{L}(\mathcal{N}_{j'}, \mathcal{N}_j)$ .

We define the polynomially localized block matrices

$$\mathcal{A}_s := \{A = (A_j^{j'})_{j, j' \in J^+} : |A|_s^2 := \sup_{j \in J^+} \sum_{j' \in J^+} e^{2s|j-j'|} \|A_j^{j'}\|_0^2 < \infty\},$$

where  $\|A_j^{j'}\|_0 := \sup_{u \in \mathcal{N}_{j'}, \|u\|_0=1} \|A_j^{j'} u\|_0$  is the  $\mathbf{L}^2$ -operator norm in  $\mathcal{L}(\mathcal{N}_{j'}, \mathcal{N}_j)$ . If  $s' > s$ , then these holds  $\mathcal{A}_{s'} \subset \mathcal{A}_s$ .

The next lemma (see [2]) shows the algebra property of  $\mathcal{A}_s$  and interpolation inequality.

**Lemma 7** *There holds*

$$|AB|_s \leq c(s)|A|_s|B|_s, \quad \forall A, B \in \mathcal{A}_s, \quad s > s_0 > \frac{r+n+1}{2}, \quad (13)$$

$$|AB|_s \leq c(s)(|A|_s|B|_{s_0} + |A|_{s_0}|B|_s), \quad s \geq s_0, \quad (14)$$

$$\|Au\|_s \leq c(s)(|A|_s\|u\|_{s_0} + |A|_{s_0}\|u\|_s), \quad \forall u \in \mathbf{H}_s, \quad s \geq s_0. \quad (15)$$

By Lemma 7, we can get,  $\forall m \in \mathbf{N}$ ,

$$|A^m|_s \leq c(s)^{m-1}|A|_s^m, \quad (16)$$

$$|A^m|_s \leq m(c(s)|A|_{s_0})^{m-1}|A|_s. \quad (17)$$

The next two lemmas can be obtained by a small modification of the proof of Lemma 2.18 and Proposition 2.19 in [2], so we omit it.

**Lemma 8** *Let  $A \in \mathcal{A}_s$ ,  $\Omega_1, \Omega_2 \subset J^+$ , and  $\Omega_1 \cap \Omega_2 = \emptyset$ . Then*

$$\|A_{\Omega_2}^{\Omega_1}\|_0 \leq c(s)|A|_s d^{-1}(\Omega_1, \Omega_2)^{2s-(r+n+1)},$$

where  $r+n+1$  is the dimension of  $J^+$ .

Since  $\mathbf{H}_s$  is an algebra, for each  $b \in \mathbf{H}^s$  defines the multiplication operator

$$u(x) \mapsto b(x)u(x), \quad \forall u \in \mathbf{H}_s, \quad (18)$$

which is represented by  $(B_j^{j'})_{j,j' \in J^+}$  with  $B_j^{j'} := \Pi_{\mathcal{N}_j} b(x)|_{\mathcal{N}_{j'}} \in \mathcal{L}(\mathcal{N}_{j'}, \mathcal{N}_j)$ .

Using Lemmas 5-6, we obtain

**Lemma 9** *If  $b \in \mathbf{H}_s$  is real, then the matrix  $(B_j^{j'})_{j,j' \in J^+}$  is self-adjoint, i.e.  $(B_j^{j'})^\dagger = (B_{j'}^j)$ , and  $\forall 2s \geq d+n+1$ ,*

$$\|B_j^{j'}\|_0 \leq c(s)\|b\|_s e^{-(s-\frac{d+n+1}{2})|j-j'|}.$$

We need to consider restricted matrices. Given a set of indexes  $l \subset J^+$ , we define

$$\mathcal{A}_s(l) := \{A = (A_j^{j'})_{j,j' \in J^+} : (A_j^{j'})^\dagger = A_j^{j'}, |A|_s^2 := \sup_{j \in l} \sum_{j' \in l} e^{2s|j-j'|} \|A_j^{j'}\|_0^2 < \infty\}.$$

The next two lemmas can be seen as the corollaries of Lemma 2.9 (see [2]).

**Lemma 10** *For real functions  $b \in \mathbf{H}_{s+s'}$  with  $2s' \geq d+r+2n+3$ , the matrix  $(B_j^{j'})_{j,j' \in J^+}$  representing the multiplication operator (18) is self-adjoint, it belongs to the algebra of polynomially localized matrices  $\mathcal{A}_s$ , and we have*

$$|B|_s \leq K(s)\|b\|_{s+s'}.$$

**Lemma 11** For  $A = (A_j^{j'})_{j,j' \in J^+} \in \mathcal{A}_s$ , its restriction  $A_l = (A_j^{j'})_{j,j' \in l} \in \mathcal{A}_s(l)$  satisfies  $|A_l|_s \leq |A|_s$ . On the other hand, any  $A \in \mathcal{A}_s(l)$  can be extended to a matrix in  $\mathcal{A}_s$  by setting  $A_j^{j'} = 0$  for  $j, j' \in l$  without changing the norm  $|A|_s$ .

Lemma 11 tells us that all the properties (algebra, interpolation, etc) hold for  $\mathcal{A}_s(l)$  with constants in dependent of  $l$ . We use  $I_l$  to denote the projectors

$$\Pi_l : \mathbf{H}_s \longrightarrow \mathbf{H}_l := \bigoplus_{j \in l \cap J^+} \mathcal{N}_j \text{ satisfy } |I_l| = 1, \forall s \geq 0.$$

### 3 Nash-Moser-type iteration scheme

Let  $(X_s, \|\cdot\|_s)_{s \geq 0}$  be a scale of Banach spaces such that

$$\forall s \leq s', \quad X_{s'} \subseteq X_s, \quad \|u\|_s \leq \|u\|_{s'}, \quad \forall u \in X_{s'}.$$

We define the finite dimensional subspaces

$$\mathbf{H}_s^{(N_i)} := \bigoplus_{j \in J_{N_i}^+} \mathcal{N}_j \subset \cap_{s \geq 0} X_s,$$

where  $J_N^+ := \{j \in J^+ | |j + \vec{\rho}| \leq N_i\}$ ,  $X_s = \mathbf{H}_s(\mathcal{M}, \mathbf{R})$ ,  $\forall s \leq k$ ,  $i$  denotes the “ $i$ ”th iterative step. For a given suitable  $N_0 > 1$ , we take  $N_i \leq N_{i+1}$  and  $N_i = N_0^i$ ,  $\forall i \in \mathbf{N}$ .

Let  $(\mathbf{H}_s^{(N_i)})_{N_i \geq 0}$  be an increasing family of closed subspaces of  $\cap_{s \geq 0} X_s$  with projectors  $\Pi^{(N_i)} : X_s \longrightarrow \mathbf{H}_s^{(N_i)}$  satisfying the “smoothing” properties:

$$\begin{aligned} \|\Pi^{(N_i)} u\|_{s+d} &\leq N_i^d \|u\|_s, \quad \forall u \in X_s, \quad \forall s, d \geq 0, \\ \|(I - \Pi^{(N_i)})u\|_s &\leq N_i^{-d} \|u\|_{s+d}, \quad \forall u \in X_{s+d}, \quad \forall s, d \geq 0. \end{aligned} \quad (19)$$

Moreover, there holds

$$\Pi^{(N_i)} u := \sum_{j \in J_{N_i}^+} \Pi_{\mathcal{N}_j} u.$$

Consider

$$L_a u = \varepsilon f(x, u), \quad \text{where } L_a := -\Delta + 1 + \varepsilon a \Delta^\varrho. \quad (20)$$

The linearized operator of (20) has the following form

$$L_a^{(N_i)} := \Pi^{(N_i)} (L_a - \varepsilon D_u f(\delta, u))|_{\mathbf{H}_s^{(N_i)}}. \quad (21)$$

Before constructing first step approximation, the invertible property of operators  $L_a^{(N_i)}$  is needed. We give the proof of the following result in next section.

**Lemma 12** *Assume that*

$$|m - an| \geq \frac{\gamma_1}{\max(1, |m|^{\frac{3}{2}})}, \quad 0 < \gamma_1 < 1, \quad \forall (m, n) \in \mathbf{Z}^2 \setminus \{(0, 0)\}, \quad (22)$$

and  $\|q\|_{\bar{\sigma}} \leq 1, \forall 1 \leq r \leq N, \forall \kappa \geq 1,$

$$\|(L_a^{(r)}(\delta, q))^{-1}\|_0 \leq \frac{4r^\kappa}{\gamma_1}. \quad (23)$$

Then the linearized operator  $L_a^{(N)}(\delta, q)$  is invertible and  $\forall s_2 > s_1 > \bar{\sigma} > 0$ , the linearized operator  $L_a^{(N)}$  satisfies

$$\|(L_a^{(N)}(\delta, q))^{-1}u\|_{s_1} \leq C(s_2 - s_1)N^{\tau+\kappa_0} (1 + \varepsilon\varsigma^{-1}\|q\|_{s_2}^p)^3 \|u\|_{s_2}, \quad (24)$$

where  $C(s_2 - s_1) = c(s_2 - s_1)^{-\tau}$ ,  $c = c(\varsigma, \tau, s, \tilde{s}, \gamma_1, \gamma)$  denotes a constant.

In fact, in the iteration process,  $N$  depends on the iteration step  $i$ . By (20), we define

$$\mathcal{J}_1(u) = L_a u - \varepsilon \Pi^{(N_i)} f(x, u) = 0. \quad (25)$$

Next we construct the “first step approximation”.

**Lemma 13** *Assume that  $a$  is diophantine. Then system (25) has the “first step approximation”  $u_1 \in \mathbf{H}_s^{(N_1)}$ :*

$$u_1 = -(L_a^{(N_1)})^{-1} E_0 \in \mathbf{H}_s^{(N_1)}, \quad (26)$$

$$E_1 = R_0 = -\varepsilon \Pi^{(N_1)} (f(x, u_0 + u_1) - f(x, u_0) - D_u f(x, u_0) u_1). \quad (27)$$

*Proof* Assume that we have chosen suitable the “0th step” approximation solution  $u_0$ . Then, the target is to get the “1th step” approximation solution.

Denote

$$E_0 = L_a u_0 - \varepsilon \Pi^{(N_1)} f(x, u_0). \quad (28)$$

By (25), we have

$$\begin{aligned} \mathcal{J}_1(u_0 + u_1) &= L_a(u_0 + u_1) - \varepsilon \Pi^{(N_1)} f(x, u_0 + u_1) \\ &= L_a u_0 - \varepsilon \Pi^{(N_1)} f(x, u_0) + L_a u_1 + \varepsilon \Pi^{(N_1)} D_u f(x, u_0) u_1 \\ &\quad - \varepsilon \Pi^{(N_1)} (f(x, u_0 + u_1) - f(x, u_0) - D_u f(x, u_0) u_1) \\ &= E_0 + L_a^{(N_1)} u_1 + R_0. \end{aligned} \quad (29)$$

Then taking

$$E_0 + L_a^{(N_1)} u_1 = 0,$$

yields

$$u_1 = -(L_a^{(N_1)})^{-1} E_0 \in \mathbf{H}_s^{(N_1)}.$$

By (29), we denote

$$\begin{aligned} E_1 &:= R_0 = \mathcal{J}_1(u_0 + u_1) \\ &= -\varepsilon \Pi^{(N_1)}(f(x, u_0 + u_1) - f(x, u_0) - D_u f(x, u_0)u_1). \end{aligned}$$

On the other hand, by (25) and (28), we can obtain

$$E_0 = -\varepsilon(I - \Pi^{(N_0)})\Pi^{(N_1)}f(x, u_0). \quad (30)$$

This completes the proof.

In order to prove the convergence of the Newton algorithm, the following KAM-style estimate is needed. For convenience, we define

$$\tilde{E}_0 := -\varepsilon \Pi^{(N_1)}f(x, u_0). \quad (31)$$

**Lemma 14** *Assume that  $a$  is diophantine. Then for any  $0 < \alpha < \sigma$ , the following estimates hold:*

$$\begin{aligned} \|u_1\|_{\sigma-\alpha} &\leq C(\alpha)(1 + \varepsilon\varsigma^{-1}\|u_0\|_\sigma^p)^3 \|\tilde{E}_0\|_{\sigma+\tau+\kappa_0}, \\ \|E_1\|_{\sigma-\alpha} &\leq C^p(\alpha)(1 + \varepsilon\varsigma^{-1}\|u_0\|_\sigma^p)^{3p} \|\tilde{E}_0\|_{\sigma+\tau+\kappa_0}^p, \end{aligned} \quad (32)$$

where  $C(\alpha)$  is defined in (33).

*Proof* Denote

$$C(\alpha) = c(\varsigma, \tau, s, \tilde{s}, \gamma_1, \gamma)\alpha^{-\tau}. \quad (33)$$

From the definition of  $u_1$  in (26), by Lemma 12, (19) and (31), we derive

$$\begin{aligned} \|u_1\|_{\sigma-\alpha} &= \|-(L_a^{(N_1)})^{-1}E_0\|_{\sigma-\alpha} \\ &\leq C(\alpha)N_1^{\tau+\kappa_0}(1 + \varepsilon\varsigma^{-1}\|u_0\|_\sigma^p)^3 \|E_0\|_\sigma \\ &\leq C(\alpha)(1 + \varepsilon\varsigma^{-1}\|u_0\|_\sigma^p)^3 \|\tilde{E}_0\|_{\sigma+\tau+\kappa_0}. \end{aligned} \quad (34)$$

By assumption (8) and the definition of  $E_1$ , we have

$$\begin{aligned} \|E_1\|_{\sigma-\alpha} &= \|\Pi^{(N_1)}(f(x, u_0 + u_1) - f(x, u_0) - D_u f(x, u_0)u_1)\|_{\sigma-\alpha} \\ &\leq \|u_1\|_{\sigma-\alpha}^p \\ &\leq C^p(\alpha)(1 + \varepsilon\varsigma^{-1}\|u_0\|_\sigma^p)^{3p} \|\tilde{E}_0\|_{\sigma+\tau+\kappa_0}^p. \end{aligned}$$

This completes the proof.

For  $i \in \mathbf{N}$  and  $0 < \sigma_0(\mathcal{M}) < \bar{\sigma}(\mathcal{M}) < \sigma(\mathcal{M}) < k(\mathcal{M}) - 1$ , set

$$\sigma_i := \bar{\sigma} + \frac{\sigma - \bar{\sigma}}{2^i}, \quad (35)$$

$$\alpha_{i+1} := \sigma_i - \sigma_{i+1} = \frac{\sigma - \bar{\sigma}}{2^{i+1}}. \quad (36)$$

By (35)-(36), it follows that

$$\sigma_0 > \sigma_1 > \dots > \sigma_i > \sigma_{i+1} > \dots, \text{ for } i \in \mathbf{N}.$$

Define

$$\begin{aligned} \mathcal{P}_1(u_0) &:= u_0 + u_1, \text{ for } u_0 \in \mathbf{H}_{\sigma_0}^{(N_0)}, \\ E_i &= \mathcal{J}_1\left(\sum_{k=0}^i u_k\right) = \mathcal{J}_1(\mathcal{P}_1^i(u_0)), \end{aligned}$$

In fact, to obtain the “ $i$  th” approximation solution  $u_i \in \mathbf{H}_{\sigma_i}^{(N_i)}$  of system (25), we need to solve following equations

$$\begin{aligned} \mathcal{J}_1\left(\sum_{k=0}^i u_k\right) &= L_a\left(\sum_{k=0}^{i-1} u_k\right) - \varepsilon \Pi^{(N_i)} f\left(x, \sum_{k=0}^{i-1} u_k\right) + L_a u_i - \varepsilon \Pi^{(N_i)} D_u f\left(x, \sum_{k=0}^{i-1} u_k\right) u_i \\ &\quad - \varepsilon \Pi^{(N_i)} \left( f\left(x, \sum_{k=0}^i u_k\right) - f\left(x, \sum_{k=0}^{i-1} u_k\right) - D_u f\left(x, \sum_{k=0}^{i-1} u_k\right) u_i \right). \end{aligned}$$

Then, we get the ‘ $i$  th’ step approximation  $u_i \in \mathbf{H}_{\sigma_i}^{(N_i)}$  :

$$u_i = -(L_a^{(N_i)})^{-1} E_{i-1}, \quad (37)$$

where

$$E_{i-1} = L_a\left(\sum_{k=0}^{i-1} u_k\right) - \varepsilon \Pi^{(N_i)} f\left(x, \sum_{k=0}^{i-1} u_k\right) = -\varepsilon (I - \Pi^{(N_{i-1})}) \Pi^{(N_i)} f\left(x, \sum_{k=0}^{i-1} u_k\right).$$

As done in Lemma 13, it is easy to get that

$$E_i := R_{i-1} = -\varepsilon \Pi^{(N_i)} \left( f\left(x, \sum_{k=0}^{i-1} u_k\right) - f\left(x, \sum_{k=0}^i u_k\right) - D_u f\left(x, \sum_{k=0}^{i-1} u_k\right) u_i \right), \quad (38)$$

$$\tilde{E}_i = -\varepsilon \Pi^{(N_i)} f\left(x, \sum_{k=0}^{i-1} u_k\right). \quad (39)$$

Hence, we only need to estimate  $R_{i-1}$  to prove the convergence of algorithm. In the following, a sufficient condition on the convergence of Newton algorithm is proved. This proof is based on Lemma 14. It also shows the existence of solutions for (25).

**Lemma 15** *Assume that  $a$  is diophantine. Then, for sufficiently small  $\varepsilon$ , equations (20) has a solution*

$$u_\infty = \sum_{k=0}^{\infty} u_k \in \mathbf{H}_{\bar{\sigma}} \cap \mathcal{B}_1(0),$$

where  $\mathcal{B}_1(0) := \{u \mid \|u\|_s \leq 1, \forall s > \bar{s} > 0\}$ .

*Proof* We divide into two cases. If  $\varepsilon\varsigma^{-1}\|u_{i-1}\|_{\sigma_{i-1}}^p < 1$ , by Lemma 12, (37) and (39), we derive

$$\begin{aligned} \|u_i\|_{\sigma_i} &= \|-(L_a^{(N_i)})^{-1}E_{i-1}\|_{\sigma_i} \\ &\leq C(\alpha_i)N_i^{\tau+\kappa_0}(1+\varepsilon\varsigma^{-1}\|u_{i-1}\|_{\sigma_{i-1}}^p)^3\|E_{i-1}\|_{\sigma_{i-1}} \\ &\leq C(\alpha_i)(1+\varepsilon\varsigma^{-1}\|u_{i-1}\|_{\sigma_{i-1}}^p)^3\|\tilde{E}_{i-1}\|_{\sigma_{i-1}+\tau+\kappa_0} \\ &\leq 2C(\alpha_i)\|\tilde{E}_{i-1}\|_{\sigma_{i-1}+\tau+\kappa_0}, \end{aligned} \quad (40)$$

where  $c(\varepsilon, \varsigma)$  is a constant depending on  $\varepsilon$  and  $\varsigma$ .

Note that  $N_i = N_0^i, \forall i \in \mathbf{N}$ . By (38)-(40) and assumption (8), we have

$$\begin{aligned} \|E_i\|_{\sigma_i} &= \varepsilon\|II^{(N_i)}(f(x, \sum_{k=0}^i u_k) - f(x, \sum_{k=0}^{i-1} u_k) - D_u f(x, \sum_{k=0}^{i-1} u_k)u_i)\|_{\sigma_i} \\ &\leq \varepsilon c(s)\|u_i\|_{\sigma_i}^p \\ &\leq \varepsilon c(s)N_i^{(\tau+\kappa_0)p}C^p(\alpha_i)\|E_{i-1}\|_{\sigma_{i-1}}^p \\ &\leq (\varepsilon c(s))^{p+1}N_i^{(\tau+\kappa_0)p}N_{i-1}^{(\tau+\kappa_0)p^2}C^p(\alpha_i)C^{p^2}(\alpha_{i-1})\|E_{i-2}\|_{\sigma_{i-2}}^{p^2} \\ &\leq \dots \\ &\leq (\varepsilon c(s))^{\sum_{k=1}^{i-1} p^k+1}N_0^{(\tau+\kappa_0)p^{i+2}}\|E_0\|_{\sigma_0}^{p^i} \prod_{k=1}^i C^{p^k}(\alpha_{i+1-k}) \\ &\leq (\varepsilon c(s))^{p^i}(\varepsilon, \varsigma)(N_0^{(\tau+\kappa_0)p^2}\|E_0\|_{\sigma_0})^{p^i} \prod_{k=1}^i C^{p^k}(\alpha_{i+1-k}) \\ &\leq (\varepsilon c(s))^{p^i}(\varepsilon, \varsigma)\|\tilde{E}_0\|_{\sigma_0+(\tau+\kappa_0)p^2}^{p^i} \prod_{k=1}^i C^{p^k}(\alpha_{i+1-k}) \\ &\leq (8^{p^2}\varepsilon c(s)c^{p^2}(\tau, \sigma, \tilde{\sigma}, \gamma_1, \gamma))\|\tilde{E}_0\|_{\sigma_0+(\tau+\kappa_0)p^2}^{p^i}. \end{aligned} \quad (41)$$

Hence, choosing small  $\varepsilon > 0$  such that

$$8^{p^2}\varepsilon c(s)c^{p^2}(\tau, \sigma, \tilde{\sigma}, \gamma_1, \gamma)\|\tilde{E}_0\|_{\sigma_0+(\tau+\kappa_0)p^2}^{p^2} = 8^{p^2}\varepsilon c(s)c^{p^2}(\tau, \sigma, \tilde{\sigma}, \gamma_1, \gamma)N_0^{(\tau+\kappa_0)p^2}\|\tilde{E}_0\|_{\sigma_0} < 1.$$

For any fixed  $p > 1$ , we derive

$$\lim_{i \rightarrow \infty} \|E_i\|_{\sigma_i} = 0. \quad (42)$$

If  $\varepsilon\varsigma^{-1}\|u_{i-1}\|_{\sigma_{i-1}}^p \geq 1$ , by Lemma 12, (37) and (39), we derive

$$\begin{aligned} \|u_i\|_{\sigma_i} &= \|-(L_a^{(N_i)})^{-1}E_{i-1}\|_{\sigma_i} \\ &\leq C(\alpha_i)N_i^{\tau+\kappa_0}(1+\varepsilon\varsigma^{-1}\|u_{i-1}\|_{\sigma_{i-1}}^p)^3\|E_{i-1}\|_{\sigma_{i-1}} \\ &\leq 2\varepsilon^3\varsigma^{-3}C(\alpha_i)\|u_{i-1}\|_{\sigma_{i-1}}^{3p}\|\tilde{E}_{i-1}\|_{\sigma_{i-1}+\tau+\kappa_0} \\ &\leq (2\varepsilon\varsigma^{-1})^{3(p+1)}C(\alpha_i)C^{3p}(\alpha_{i-1})\|u_{i-2}\|_{\sigma_{i-2}}^{(3p)^2}\|\tilde{E}_{i-2}\|_{\sigma_{i-2}+\tau+\kappa_0}^{3p}\|\tilde{E}_{i-1}\|_{\sigma_{i-1}+\tau+\kappa_0} \\ &\leq \dots \\ &\leq (2\varepsilon\varsigma^{-1})^{\sum_{k=0}^{i-1} (3p)^k}\|u_0\|_{\sigma_0}^{(3p)^i} \prod_{k=1}^i C^{(3p)^{k-1}}(\alpha_{i+1-k})\|\tilde{E}_{i-k}\|_{\sigma_{i-k}+\tau+\kappa_0}^{(3p)^{k-1}}. \end{aligned} \quad (43)$$



But we will choose the initial step  $u_0 = 0$  in this paper, which combining with (43) leads to  $\|u_i\|_{\sigma_i} = 0$ ,  $\forall i \in \mathbf{N}$ . This contradicts with assumption  $\varepsilon\varsigma^{-1}\|u_{i-1}\|_{\sigma_{i-1}}^p > 1$ . Hence, the case is not possible. (20) has a solution

$$u_\infty := \sum_{k=0}^{\infty} u_k \in \mathbf{H}_{\bar{\sigma}} \cap \mathcal{B}_1(0),$$

where  $\mathcal{B}_1(0) := \{u \mid \|u\|_s \leq 1, \forall s > \bar{s} > 0\}$ . This completes the proof.

Next result gives the local uniqueness of solutions for equation (20).

**Lemma 16** *Assume that  $a$  is diophantine. Equation (20) has a unique solution  $u \in \mathbf{H}_{\bar{\sigma}} \cap \mathcal{B}_1(0)$  obtained in Lemma 15.*

*Proof* Let  $u, \tilde{u} \in \mathbf{H}_{\bar{\sigma}} \cap \mathcal{B}_1(0)$  be two solutions of system (25), where

$$\mathbf{B}_1(0) := \{u \mid \|u\|_s < \delta, \text{ for some } \delta < 1, \forall s > \sigma_0\}.$$

Write  $h = u - \tilde{u}$ . Our target is to prove  $h = 0$ . By (25), we have

$$L_a h - \varepsilon \Pi^{(N_i)} D_u f(x, u) h - \varepsilon \Pi^{(N_i)} (f(x, u) - f(x, \tilde{u}) - D_u f(x, u) h) = 0,$$

which implies that

$$h = \varepsilon (L_a - \varepsilon \Pi^{(N_i)} D_u f(x, u))^{-1} \Pi^{(N_i)} (f(x, u) - f(x, \tilde{u}) - D_u f(x, u) h). \quad (44)$$

Note that  $N_i = N_0^i$ ,  $\forall i \in \mathbf{N}$ . Thus, by Lemma 12 and (44), we have

$$\begin{aligned} \|h\|_{\sigma_i} &= \varepsilon \|(L_a^{N_i})^{-1} \Pi^{(N_i)} (f(x, u) - f(x, \tilde{u}) - D_u f(x, u) h)\|_{\sigma_i} \\ &\leq C(\alpha_i) N_i^{\tau+\kappa_0} (1 + \varepsilon\varsigma^{-1} \|u\|_{\sigma_{i-1}}^p) \|h\|_{\sigma_{i-1}}^p \\ &\leq 2^{p+1} N_i^{(\tau+\kappa_0)} N_{i-1}^{(\tau+\kappa_0)p} C(\alpha_i) C^p(\alpha_{i-1}) \|h\|_{\sigma_{i-2}}^p \\ &\leq \dots \\ &\leq 2^{\sum_{k=0}^{i-1} p^k} N_0^{(\tau+\kappa_0)(\sum_{k=0}^{i-1} p^k)} \|h\|_{\sigma_0}^i \prod_{k=1}^i C^{p^{k-1}}(\alpha_{i+1-k}) \\ &\leq (8^{p^2} c^{p^2} (\varepsilon, \varsigma, \tau, s, \tilde{s}, \gamma_1, \gamma) N_0^{(\tau+\kappa_0)p} \|h\|_{\sigma_0})^{p^i}. \end{aligned}$$

Choosing  $\delta < 8^{-p^2} c^{-p^2} (\varepsilon, \varsigma, \tau, s, \tilde{s}, \gamma_1, \gamma) N_0^{-(\tau+\kappa_0)p}$ , we obtain

$$\lim_{i \rightarrow \infty} \|h\|_{\bar{\sigma}} = 0.$$

This completes the proof.

*Remark 1* The dependence upon the parameter, as is well known, is more delicate since it involves in the small divisors of  $\omega_j$ : it is, however, standard to check that this dependence is  $\mathbf{C}^1$  on a bounded set of Diophantine numbers, for more details, see, for example, [1, 2].

By Lemma 12, for sufficient small  $\delta_0 > 0$  and given  $r > 0$ , we define

$$\begin{aligned} Y_{\gamma_1, \kappa_0}^{(N)} &:= \{(\delta, q) \in [0, \delta_0) \times \mathbf{H}^{(N)} \mid \|q\|_{\bar{\sigma}} \leq 1, \varepsilon\delta \text{ satisfies (22) - (23)}\}, \\ U_r^{(N)} &:= \{u \in \mathbf{C}^1([0, \delta_0), \mathbf{H}^N) \mid \|u\|_{\bar{\sigma}} \leq 1, \|\partial_\delta u\|_{\bar{\sigma}} \leq r\}, \\ \mathcal{G}_{\gamma_1, \kappa_0}^{(N)} &:= \{\delta \in [0, \delta_0) \mid (\delta, u(\delta)) \in Y_{\gamma_1, \kappa_0}^{(N)} \text{ and } u \in U_r^{(N)}\}, \\ \mathcal{G}_r &:= \{\delta \in [0, \delta_0) \mid \|(L_a^{(r)}(\delta, q(\delta)))^{-1}\| \leq \frac{4r^\kappa}{\gamma_1}\}, \\ \mathcal{G} &:= \{\delta \in [0, \delta_0) \mid \omega(\delta) \text{ satisfies (22)}\}. \end{aligned}$$

Then for a given function  $\delta \mapsto q(\delta) \in U_r^{(N)}$ , the set  $\mathcal{G}_{\gamma_1, \kappa_0}^{(N)}$  is equivalent to

$$\mathcal{G}_{\gamma_1, \kappa_0}^{(N)} = \cap_{1 \leq r \leq N} \mathcal{G}_r \cap \mathcal{G}.$$

Choosing  $\kappa$  and  $\gamma_1$  such that

$$\kappa \geq \max\{\tau, 2 + d + n + \frac{2\rho - 2}{2\rho - 1}(\tau + 2\rho)\}, \quad \gamma_1 \in (0, \gamma_2], \text{ for } \gamma_2 \leq \gamma_1. \quad (45)$$

Next we have the measure estimate. The proof of it will be given in Appendix.

**Lemma 17** (*Measure estimates*) Assume that  $\varepsilon$  is diophantine,  $\varepsilon_0 \gamma^{-1} M^{\tau+2e}$  is sufficient small and (45) holds. Then  $\mathcal{G}_{\gamma_1, \kappa_0}^{(M)}(0) = \mathcal{G}$ , and  $\mathcal{G}$  satisfies

$$|(\mathcal{G}_{\gamma_1, \kappa_0}^{(M)}(0))^c \cap [0, \delta]| \leq C\gamma_1 \delta, \quad \forall \delta \in (0, \delta_0]. \quad (46)$$

Furthermore, for any  $r' > 0$ , there exists  $\delta' := \delta'(\gamma_1, r')$  such that the measure estimate

$$|(\mathcal{G}_{\gamma_1, \kappa_0}^{(N')}(u_2))^c \setminus (\mathcal{G}_{\gamma_1, \kappa_0}^{(N)}(u_1))^c \cap [0, \delta]| \leq C\gamma_1 \delta N^{-1}, \quad \forall \delta \in (0, \delta'] \quad (47)$$

holds, where  $N' \geq N \geq M$ ,  $u_1 \in U_{r'}^{(N)}$ ,  $u_1 \in U_{r'}^{(N)}$  with  $\|u_2 - u_1\|_{\bar{\sigma}} \leq N^{-e}$ ,  $e$  denotes a constant depending on  $\kappa_0$  and  $n$ .

#### 4 Proof of Lemma 12

This section is devoted to give the proof of Lemma 12. Let

$$b(x) := (\partial_u f)(\delta, u).$$

For notational convenience, we denote  $N = N_i$ . Due to the orthogonal decomposition  $\mathbf{H}^{(N)} = \bigoplus_{j \in J_N^+} \mathcal{N}_j$ , we define

$$h \mapsto \mathbf{L}^{(N)}[h] := \Pi^{(N)}(L_a h - \varepsilon b(x)h), \quad \forall h \in \mathbf{H}^{(N)}. \quad (48)$$

We write (48) by the block matrix

$$L_a^{(N)} = D + \varepsilon T, \quad D := \text{diag}_{j \in J_N^+}(D_j I_j), \quad (49)$$

where  $j = (j_1, j_2) \in \Lambda^+ \times \mathbf{Z}^n$ ,

$$D_j := |j_1 + \rho|^2 - |\rho|^2 + |j_2|^2 + 1 - \varepsilon a(|j_1 + \rho|^2 - |\rho|^2 + |j_2|^2)^e, \quad (50)$$

$I_j$  is the identity map in  $\mathcal{N}_j$ , and

$$T := (T_j^{j'})_{j, j' \in J_N^+}, \quad T_j^{j'} := \Pi_{\mathcal{N}_j} L_a^{(N)}|_{\mathcal{N}_{j'}} \in \mathcal{L}(\mathcal{N}_{j'}, \mathcal{N}_j). \quad (51)$$

In what follows, we prove the estimate (24). For fixing  $\varsigma > 0$ , we define the regular sites  $R$  and the singular sites  $S$  as

$$R := \{j \in J^+ \mid |D_j| \geq \varsigma\} \quad \text{and} \quad S := \{j \in J^+ \mid |D_j| < \varsigma\}. \quad (52)$$

For each  $N$ , we denote the restrictions of  $S$ ,  $R$ ,  $\Omega_\alpha$  to  $J_N^+$  with the same symbols. The following result shows the separation of singular sites, and the proof will be completed in the appendix.

**Lemma 18** *Assume that  $a$  is diophantine and  $a$  satisfies (22). There exists  $\varsigma_0(\gamma)$  such that for  $\varsigma \in (0, \varsigma_0(\gamma)]$  and a partition of the singular sites  $S$  which can be partitioned in pairwise disjoint clusters  $\Omega_\alpha$  as*

$$S = \bigcup_{\alpha \in N} \Omega_\alpha \quad (53)$$

satisfying

- (dyadic)  $\forall \alpha, M_\alpha \leq 2m_\alpha$ , where  $M_\alpha := \max_{j \in \Omega_\alpha} |j + \vec{\rho}|$ ,  $m_\alpha := \max_{j \in \Omega_\alpha} |j + \vec{\rho}|$ .
- (separation)  $\exists \lambda, c > 0$  such that  $d(\Omega_\alpha, \Omega_\beta) \geq c(M_\alpha + M_\beta)^\lambda$ ,  $\forall \alpha \neq \beta$ , where  $d(\Omega_\alpha, \Omega_\beta) := \max_{j \in \Omega_\alpha, j' \in \Omega_\beta} |j - j'|$  and  $\lambda$  depends only on  $\mathcal{M}$ .

Using Lemma 10, we have the following.

**Lemma 19** *Let  $2s' \geq d + r + 2n + 3$ . For a real  $b \in \mathbf{H}_{s+s'}$ , the matrix  $T = (T_j^{j'})_{j, j' \in J_N^+}$  defined in (51) is self-adjoint and belongs to the algebra of polynomially localized matrices  $\mathcal{A}_s(J_N^+)$  with*

$$|T|_s \leq K(s) \|b\|_{s+s'}.$$

Moreover, for any  $s > s'$ ,

$$|T|_s \leq K'(s) N^{s'} \|b\|_s.$$

Since the decomposition

$$\mathbf{H}^{(N)} := \mathbf{H}_R \oplus \mathbf{H}_S,$$

with

$$\mathbf{H}_R := \bigoplus_{j \in R \cap J_N^+} \mathcal{N}_j, \quad \mathbf{H}_S := \bigoplus_{j \in S \cap J_N^+} \mathcal{N}_j,$$

we can represent the operator  $L_a^{(N)}$  as the self-adjoint block matrix

$$L_a^{(N)} = \begin{pmatrix} L_R & L_R^S \\ L_R^R & L_S \end{pmatrix},$$

where  $L_R^S = (L_S^R)^\dagger$ ,  $L_R = L_R^\dagger$ ,  $L_S = L_S^\dagger$ .

Thus the invertibility of  $L_a^{(N)}$  can be expressed via the "resolvent-type" identity

$$(L_a^{(N)})^{-1} = \begin{pmatrix} I & -L_R^{-1}L_R^S \\ 0 & I \end{pmatrix} \begin{pmatrix} L_R^{-1} & 0 \\ 0 & \mathcal{L}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -L_R^R L_R^{-1} & I \end{pmatrix}, \quad (54)$$

where the "quasi-singular" matrix

$$\mathcal{L} := L_S - L_S^R L_R^{-1} L_R^S \in \mathcal{A}_s(S).$$

The reason of  $\mathcal{L} \in \mathcal{A}_s(S)$  is that  $\mathcal{L}$  is the restriction to  $S$  of the polynomially localized matrix

$$I_S(L - I_S L I_R \tilde{L}^{-1} I_R L I_S) I_S \in \mathcal{A}_s,$$

where

$$\tilde{L}^{-1} = \begin{pmatrix} I & 0 \\ 0 & L_R \end{pmatrix}.$$

**Lemma 20** *Assume that nonresonance condition (6) holds. For  $s_0 < s_1 < s_2 < k-1$ ,  $|L_R^{-1}|_{s_0} \leq 2\varsigma^{-1}$ , the operator  $L_R$  satisfies*

$$|\tilde{L}_R^{-1}|_{s_1} \leq c(s_1)(1 + \varepsilon\varsigma^{-1}|T|_{s_1}), \quad (55)$$

$$\|L_R^{-1}u\|_{s_1} \leq c(\gamma, \tau, s_2)(s_2 - s_1)^{-\tau}(1 + \varepsilon\varsigma^{-1}|T|_{s_2})\|u\|_{s_2}, \quad (56)$$

where  $\tilde{L}^{-1} = L_R^{-1}D_R$ ,  $c(\gamma, \tau, s_2)$  is a constant depending on  $\gamma, \tau, s_2$ .

*Proof* It follows from (49) and (52) that  $D_R$  is a diagonal matrix and satisfies  $|D_R^{-1}|_s \leq \varsigma^{-1}$ . By (13), we have that the Neumann series

$$\tilde{L}_R^{-1} = L_R^{-1}D_R = \sum_{m \geq 0} (-\varepsilon)^m (D_R^{-1}T_R)^m \quad (57)$$

is totally convergent in  $|\cdot|_{s_1}$  with  $|L_R^{-1}|_{s_0} \leq 2\varsigma^{-1}$ , by taking  $\varepsilon\varsigma^{-1}|T|_{s_0} \leq c(s_0)$  small enough.

Using (13) and (17), we have that  $\forall m \in \mathbf{N}$ ,

$$\begin{aligned} \varepsilon^m |(D_R^{-1}T_R)^m|_{s_1} &\leq \varepsilon^m c(s) |(D_R^{-1}T_R)^m|_{s_1} \\ &\leq c(s) \varepsilon^m m(c(s) |D_R^{-1}T_R|_{s_0})^{m-1} |D_R^{-1}T_R|_{s_1} \\ &\leq c'(s) \varepsilon m \varsigma^{-1} (\varepsilon c(s_1) \varsigma^{-1} |T|_{s_0})^{m-1} |T|_{s_1}, \end{aligned}$$

which together with (57) implies that for  $\varepsilon\zeta^{-1}|T|_{s_0} < c(s_0)$  small enough, (55) holds.

By nonresonance condition (6) and  $\sup_{x>0}(x^y e^{-x}) = (ye^{-1})^y$ ,  $\forall y \geq 0$ , we derive

$$\begin{aligned} e^{-2|j+\vec{p}|(s_2-s_1)}|\omega_j^2 + 1 - \varepsilon a\omega_j^{2p}|^{-2} &\leq \gamma^{-1}|j + \vec{p}|^\tau e^{-2|j+\vec{p}|(s_2-s_1)} \\ &\leq c(\gamma, \tau)(s_2 - s_1)^{-2\tau}. \end{aligned} \quad (58)$$

Then by (58), for any  $u \in \mathbf{H}_R$ ,

$$\begin{aligned} \|L_R^{-1}u\|_{s_1}^2 &= \sum_{j \in R \cap J_N^+} e^{2|j+\vec{p}|s_1} \|L_R^{-1}u_j\|_{\mathbf{L}^2}^2 \\ &\leq \sum_{j \in R \cap J_N^+} e^{2|j+\vec{p}|s_1} |\omega_j^2 + 1 - \varepsilon a\omega_j^{2p}|^{-2} \|\tilde{L}_R^{-1}u_j\|_{\mathbf{L}^2}^2 \\ &\leq \sum_{j \in R \cap J_N^+} e^{-2|j+\vec{p}|(s_2-s_1)} |\omega_j^2 + 1 - \varepsilon a\omega_j^{2p}|^{-2} e^{2|j+\vec{p}|s_2} \|\tilde{L}_R^{-1}u_j\|_{\mathbf{L}^2}^2 \\ &\leq c(\gamma, \tau)(s_2 - s_1)^{-2\tau} \|\tilde{L}_R^{-1}u\|_{s_2}^2. \end{aligned}$$

Thus using interpolation (15) and (55), we derive that for  $s_1 < s < s_2$ ,

$$\begin{aligned} \|L_R^{-1}u\|_{s_1} &\leq c(\gamma, \tau)(s_2 - s_1)^{-\tau} \|\tilde{L}_R^{-1}u\|_{s_2} \\ &\leq c(r, \tau, s_2)(s_2 - s_1)^\tau (|\tilde{L}_R^{-1}|_{s_2} \|u\|_s + |\tilde{L}_R^{-1}|_s \|u\|_{s_2}) \\ &\leq c(r, \tau, s_2)(s_2 - s_1)^\tau (1 + \varepsilon\zeta^{-1}|T|_{s_2}) \|u\|_{s_2}. \end{aligned}$$

This completes the proof.

Next we analyse the quasi-singular matrix  $\mathcal{L}$ . By (53), the singular sites restricted to  $J_N^+$  are

$$S = \bigcup_{\alpha \in l_N} \Omega_\alpha, \quad \text{where } l_N := \{\alpha \in \mathbf{N} | m_\alpha \leq N\},$$

and  $\Omega_\alpha \equiv \Omega_\alpha \cup J_N^+$ . Due to the decomposition  $\tilde{H}_S := \bigoplus_{\alpha \in l_N} \tilde{H}_\alpha$ , where  $\mathbf{H}_\alpha := \bigoplus_{j \in \Omega_\alpha} \mathcal{N}_j$ , we represent  $\mathcal{L}$  as the block matrix  $\mathcal{L} = (\mathcal{L}_\alpha^\beta)_{\alpha, \beta \in l_N}$ , where  $\mathcal{L}_\alpha^\beta := \Pi_{\mathbf{H}_\alpha} \mathcal{L}|_{\mathbf{H}_\beta}$ . So we can rewrite

$$\mathcal{L} = \mathcal{D} + \mathcal{T},$$

where  $\mathcal{D} := \text{diag}_{\alpha \in l_N}(\mathcal{L}_\alpha)$ ,  $\mathcal{L}_\alpha := \mathcal{L}_\alpha^\alpha$ ,  $\mathcal{T} := (\mathcal{L}_\alpha^\beta)_{\alpha \neq \beta}$ .

We define a diagonal matrix corresponding to the matrix  $\mathcal{D}$  as  $\bar{D} := \text{diag}_{\alpha \in l_N}(\bar{L}_\alpha)$ , where  $\bar{L}_\alpha = \text{diag}_{j \in \Omega_\alpha}(D_j)$ .

To show  $\mathcal{D}$  is invertible, we only need to prove that  $\mathcal{L}_\alpha$  is invertible,  $\forall \alpha \in l_N$ .

**Lemma 21**  $\forall \alpha \in l_N$ ,  $\mathcal{L}_\alpha$  is invertible and  $\|\mathcal{L}_\alpha^{-1}\|_0 \leq C\gamma_1^{-1}M_\alpha^\kappa$ .

The proof process of above Lemma is similar with Lemma 6.6 in [2], so we omit it.

**Lemma 22** *Assume that nonreonance condition (6) holds. We have*

$$\|\mathcal{D}^{-1}\bar{D}u\|_{s_1} \leq c(\varsigma, s_1, \gamma_1)N^\tau\|u\|_{s_2},$$

where  $c(\varsigma, s_1, \gamma_1)$  is a constant which depends on  $\varsigma$ ,  $s_1$  and  $\gamma_1$ .

*Proof* Note that  $\|u_\alpha\|_0 \leq m_\alpha^{-s_1}\|u_\alpha\|_{s_1}$  and  $M_\alpha = 2m_\alpha$ . So for any  $u = \sum_{\alpha \in l_N} u_\alpha \in \mathbf{H}_\alpha$ ,  $u_\alpha \in \mathbf{H}_\alpha$ ,

$$\begin{aligned} \|\mathcal{D}^{-1}\bar{D}u\|_{s_1}^2 &= \sum_{\alpha \in l_N} \|\mathcal{L}_\alpha^{-1}\bar{L}_\alpha u_\alpha\|_{s_1}^2 \leq \sum_{\alpha \in l_N} M_\alpha^{2s_1} \|\mathcal{L}_\alpha^{-1}\bar{L}_\alpha u_\alpha\|_0^2 \\ &\leq c\gamma_1^{-2} \sum_{\alpha \in l_N} M_\alpha^{2(s_1+\tau)} \|\bar{L}_\alpha u_\alpha\|_0^2 \\ &\leq c\gamma_1^{-2} \sum_{\alpha \in l_N} M_\alpha^{2(s_1+\tau)} m_\alpha^{-2s_1} \|\bar{L}_\alpha u_\alpha\|_{s_1}^2 \\ &\leq c\gamma_1^{-2} 4^{s_1} \sum_{\alpha \in l_N} M_\alpha^{2\tau} \|\bar{L}_\alpha u_\alpha\|_{s_1}^2 \\ &\leq c\gamma_1^{-2} 4^{s_1} N^{2\tau} \sum_{\alpha \in l_N} \|\bar{L}_\alpha u_\alpha\|_{s_1}^2 \\ &= c\gamma_1^{-2} 4^{s_1} N^{2\tau} \|\bar{D}u\|_{s_1}^2. \end{aligned} \quad (59)$$

Using interpolation (15) and (52), for  $0 < s_1 < s_2$ , it follows from (59) that

$$\begin{aligned} \|\mathcal{D}^{-1}\bar{D}u\|_{s_1} &\leq c\gamma_1^{-1} 2^{s_1} N^\tau \|\bar{D}u\|_{s_1} \\ &\leq c\gamma_1^{-1} 2^{s_1} N^\tau (|\bar{D}|_{s_2} \|u\|_{s_1} + |\bar{D}|_{s_1} \|u\|_{s_2}) \\ &\leq c(\varsigma) \gamma_1^{-1} 2^{s_1+1} N^\tau \|u\|_{s_2}. \end{aligned}$$

This completes the proof.

The following result is taken from [2], so we omit the proof.

**Lemma 23** *For  $\kappa_0 = \tau + r + n + 1$ ,  $\forall s \geq 0$ ,  $\forall m \in \mathbf{N}$ , there hold:*

$$c(s_1) \|\mathcal{D}^{-1}\mathcal{T}\|_{s_0} < \frac{1}{2}, \quad \|\mathcal{D}^{-1}\|_s \leq c(s) \gamma_1^{-1} N^\tau, \quad (60)$$

$$\|(\mathcal{D}^{-1}\mathcal{T})^m u\|_s \leq (\varepsilon \gamma^{-1} K(s))^m (m N^{\kappa_0} |T|_s |T|_{s_0}^{m-1} \|u\|_{s_0} + |T|_{s_0}^m \|u\|_s). \quad (61)$$

**Lemma 24** *Assume that nonreonance condition (6) holds. For  $0 < s_0 < s_1 < s_2 < s_3 < k - 1$ , we have*

$$\|\mathcal{L}^{-1}u\|_{s_1} \leq c(\varsigma, \tau, s_1, \gamma_1, \gamma) N^{\tau+\kappa_0} (s_3 - s_2)^{-\tau} (\|u\|_{s_3} + \varepsilon |T|_{s_1} \|u\|_{s_2}). \quad (62)$$

*Proof* The Neumann series

$$\mathcal{L}^{-1} = (I + \mathcal{D}^{-1}\mathcal{T})^{-1}\mathcal{D}^{-1} = \sum_{m \geq 0} (-1)^m (\mathcal{D}^{-1}\mathcal{T})^m \mathcal{D}^{-1} \quad (63)$$

is totally convergent in operator norm  $\|\cdot\|_{s_0}$  with  $\|\mathcal{L}^{-1}\|_{s_0} \leq c\gamma_1^{-1}N^\tau$ , by using (60).

By (61) and (63), we have

$$\begin{aligned} \|\mathcal{L}^{-1}u\|_{s_1} &\leq \|\mathcal{D}^{-1}u\|_{s_1} + \sum_{m \geq 1} \|(\mathcal{D}^{-1}\mathcal{T})^m \mathcal{D}^{-1}u\|_{s_1} \\ &\leq \|\mathcal{D}^{-1}u\|_{s_1} + \|\mathcal{D}^{-1}u\|_{s_1} \sum_{m \geq 1} (\varepsilon\gamma_1^{-1}K(s)|T|_{s_0})^m \\ &\quad + N^{\kappa_0}K(s_1)\varepsilon\gamma_1^{-1}|T|_{s_1}\|\mathcal{D}^{-1}u\|_{s_0} \sum_{m \geq 1} m(K(s)\varepsilon\gamma_1^{-1}|T|_{s_0})^{m-1}. \end{aligned} \quad (64)$$

Using  $\sup_{x>0}(x^ye^{-x}) = (ye^{-1})^y$ ,  $\forall y \geq 0$ , for  $0 < s_1 < s_2 < s_3$ , it follows from Lemma 20 that

$$\begin{aligned} \|\mathcal{D}^{-1}u\|_{s_1}^2 &= \|\mathcal{D}^{-1}\bar{D}\bar{D}^{-1}u\|_{s_1}^2 \leq c^2(\varsigma, s_1, \gamma_1)N^{2\tau}\|\bar{D}^{-1}u\|_{s_2}^2 \\ &= c^2(\varsigma, s_1, \gamma_1)N^{2\tau} \sum_{j \in S \cap J_N^+} e^{2|j+\vec{p}|s_2}\|\bar{D}^{-1}u_j\|_{\mathbf{L}^2}^2 \\ &\leq c^2(\varsigma, s_1, \gamma_1)N^{2\tau} \sum_{j \in S \cap J_N^+} e^{2|j+\vec{p}|s_2}|\omega_j^2 + 1 - \varepsilon a\omega_j^{2p}|^{-2}\|u_j\|_{\mathbf{L}^2}^2 \\ &\leq c^2(\varsigma, s_1, \gamma_1)N^{2\tau} \sum_{j \in S \cap J_N^+} e^{-2|j+\vec{p}|(s_3-s_2)}|j+\vec{p}|^{-2}e^{2|j+\vec{p}|s_3}\|u_j\|_{\mathbf{L}^2}^2 \\ &\leq c^2(\varsigma, \tau, s_1, \gamma_1, \gamma)N^{2\tau}(s_3 - s_2)^{-2\tau}\|u\|_{s_3}^2. \end{aligned} \quad (65)$$

Thus by (64) and (65), we derive

$$\begin{aligned} \|\mathcal{L}^{-1}u\|_{s_1} &\leq \gamma_1^{-1}N^{\kappa_0}K'(s_1)(\|\mathcal{D}^{-1}u\|_{s_1} + \varepsilon|T|_{s_1}\|\mathcal{D}^{-1}u\|_{s_0}) \\ &\leq c(\varsigma, \tau, s_1, \gamma_1, \gamma)N^{\tau+\kappa_0}(s_3 - s_2)^{-\tau}(\|u\|_{s_3} + \varepsilon|T|_{s_1}\|u\|_{s_2}), \end{aligned} \quad (66)$$

where  $0 < s_1 < s_2 < s_3$  and  $\varepsilon\gamma_1^{-1}\varsigma^{-1}(1 + |T|_{s_0}) \leq c(k)$  small enough.

Now we are ready to prove Lemma 12. Let  $u = u_R + u_S$  with  $u_S \in \mathbf{H}_S$ ,  $u_R \in \mathbf{H}_R$ . Then by the resolvent identity (54),

$$\begin{aligned} \|(L^{(N)})^{-1}u\|_{s_1} &\leq \|L_R^{-1}u_R + L_R^{-1}L_S^R\mathcal{L}^{-1}(u_S + L_R^S L_R^{-1}u_R)\|_{s_1} + \|\mathcal{L}^{-1}(u_R + L_R^S L_R^{-1}u_R)\|_{s_1} \\ &\leq \|L_R^{-1}u_R\|_{s_1} + \|L_R^{-1}L_S^R\mathcal{L}^{-1}u_S\|_{s_1} + \|L_R^{-1}L_S^R\mathcal{L}^{-1}L_R^S L_R^{-1}u_R\|_{s_1} \\ &\quad + \|\mathcal{L}^{-1}u_R\|_{s_1} + \|\mathcal{L}^{-1}L_R^S L_R^{-1}u_R\|_{s_1}. \end{aligned} \quad (67)$$

Next we estimate the right hand side of (67) one by one. Using (15), (56) and (62), for  $0 < s_1 < s_2 < s_3 < s_4 < k-1$ , we have

$$\begin{aligned} \|L_R^{-1}L_S^R\mathcal{L}^{-1}u_S\|_{s_1} &\leq c(\gamma, \tau, s_2)(s_2 - s_1)^{-\tau}(1 + \varepsilon\varsigma^{-1}|T|_{s_2})\|L_S^R\mathcal{L}^{-1}u_S\|_{s_2} \\ &\leq c(\gamma, \tau, s_2)(s_2 - s_1)^{-\tau}(1 + \varepsilon\varsigma^{-1}|T|_{s_2})|T|_{s_2}\|\mathcal{L}^{-1}u\|_{s_2} \\ &\leq c(\gamma, \gamma_1, \varsigma, \tau, s_2)(s_2 - s_1)^{-\tau}(s_4 - s_3)^{-\tau}N^{\tau+\kappa_0} \\ &\quad \times (1 + \varepsilon\varsigma^{-1}|T|_{s_2})|T|_{s_2}(\|u\|_{s_3} + \varepsilon|T|_{s_2}\|u\|_{s_4}), \end{aligned} \quad (68)$$

$$\begin{aligned} \|\mathcal{L}^{-1}L_R^SL_R^{-1}u_R\|_{s_1} &\leq c(\varsigma, \tau, s_1, \gamma_1, \gamma)N^{\tau+\kappa_0}(s_3 - s_2)^{-\tau} \\ &\quad \times (\|L_R^SL_R^{-1}u_R\|_{s_3} + \varepsilon|T|_{s_1}\|L_R^SL_R^{-1}u_R\|_{s_2}) \\ &\leq c(\varsigma, \tau, s_1, s_2, s_3, \gamma_1, \gamma)N^{\tau+\kappa_0}(s_3 - s_2)^{-\tau} \\ &\quad \times (|T|_{s_3}\|L_R^{-1}u_R\|_{s_3} + \varepsilon|T|_{s_1}|T|_{s_2}\|L_R^{-1}u_R\|_{s_2}) \\ &\leq c(\varsigma, \tau, s_1, s_2, s_3, \gamma_1, \gamma)N^{\tau+\kappa_0}(s_3 - s_2)^{-\tau} \\ &\quad \times (|T|_{s_3}(s_4 - s_3)^{-\tau}(1 + \varepsilon\varsigma^{-1}|T|_{s_4})\|u\|_{s_4} \\ &\quad + \varepsilon|T|_{s_1}|T|_{s_2}(s_3 - s_2)^{-\tau}(1 + \varepsilon\varsigma^{-1}|T|_{s_3})\|u\|_{s_3}) \\ &\leq c(\varsigma, \tau, s_1, s_2, s_3, \gamma_1, \gamma)N^{\tau+\kappa_0}(s_3 - s_2)^{-\tau}|T|_{s_3}(1 + \varepsilon\varsigma^{-1}|T|_{s_4}) \\ &\quad \times ((s_4 - s_3)^{-\tau}\|u\|_{s_4} + \varepsilon|T|_{s_2}(s_3 - s_2)^{-\tau}\|u\|_{s_3}), \end{aligned} \quad (69)$$

$$\begin{aligned} \|\mathcal{L}^{-1}L_S^RL_S^{-1}L_R^SL_R^{-1}u_R\|_{s_1} &\leq c(\gamma, \tau, s_2)(s_2 - s_1)^{-\tau}(1 + \varepsilon\varsigma^{-1}|T|_{s_2})\|L_S^RL_S^{-1}L_R^SL_R^{-1}u_R\|_{s_2} \\ &\leq c(\gamma, \tau, s_2)(s_2 - s_1)^{-\tau}(1 + \varepsilon\varsigma^{-1}|T|_{s_2})|T|_{s_2}\|\mathcal{L}^{-1}L_R^SL_R^{-1}u_R\|_{s_2} \\ &\leq c(\varsigma, \tau, s_1, s_2, s_3, \gamma_1, \gamma)N^{\tau+\kappa_0}(s_3 - s_2)^{-\tau}(s_2 - s_1)^{-\tau}|T|_{s_3}^2 \\ &\quad \times (1 + \varepsilon\varsigma^{-1}|T|_{s_4})^2((s_4 - s_3)^{-\tau}\|u\|_{s_4} \\ &\quad + \varepsilon|T|_{s_2}(s_3 - s_2)^{-\tau}\|u\|_{s_3}). \end{aligned} \quad (70)$$

The terms  $\|L_R^{-1}u_R\|_{s_1}$  and  $\|\mathcal{L}^{-1}u_R\|_{s_1}$  can be controlled by using (56) and (62). Thus by (67)-(70), for  $0 < s < \tilde{s}$ , we conclude

$$\|(L^{(N)})^{-1}u\|_s \leq c(\varsigma, \tau, s, \tilde{s}, \gamma_1, \gamma)N^{\tau+\kappa_0}(1 + \varepsilon\varsigma^{-1}|T|_{\tilde{s}})^3(\tilde{s} - s)^{-\tau}\|u\|_{\tilde{s}},$$

which together with Lemma 18 gives (24).

## 5 Appendix

For completeness, we give the proof of Lemma 17 (Measure estimates) and Lemma 18, which follows essentially the scheme of [1, 2, 4].

**Proof of Lemma 17.** Note that  $|j + \vec{p}| \leq r$  and the eigenvalue of the operator  $L_a^{(r)}$  has the form  $\omega_j^2 + 1 - \varepsilon a \omega_j^{2\varrho} - O(\varepsilon)$  of the operator  $L_a^{(r)}$ . Here  $j = (j_1, j_2) \in \Lambda^+ \times \mathbf{Z}^n$ . For sufficient small  $\varepsilon_0 \gamma^{-1} M^{\tau+2\varrho}$ , by (6), all the eigenvalues of  $L_a^{(r)}$  has modulus  $\geq \gamma(4r^\tau)^{-1} \geq \gamma_1(4r^\kappa)^{-1}$ . Thus  $\mathcal{G}_r = [0, \delta_0]$  and the measure estimate (46) for  $\mathcal{G}$  is standard. To prove the measure estimate (47), we divide the process of proof into two cases. For the case  $N, N' \leq N_{\varepsilon_0} :=$



$(c\gamma_1\varepsilon_0^{-1})^{\frac{1}{\tau+2\theta}}$ ,  $\mathcal{G}_{\gamma_1,\kappa_0}^{(N')}(u_2) = \mathcal{G}_{\gamma_1,\kappa_0}^{(N)}(u_1) = \mathcal{G}$ , by the same process of proof of (46), one can prove (47) holds. For other cases, it is sufficient to prove

$$|(\mathcal{G}_{\gamma_1,\kappa_0}^{(N')}(u_2))^c \setminus (\mathcal{G}_{\gamma_1,\kappa_0}^{(N)}(u_1))^c \cap [\frac{\delta_1}{2}, \delta_1]| \leq C\gamma_1\delta N^{-1}, \quad \forall \delta_1 \in [0, \delta_0].$$

For fixed  $\delta_1$  and the decomposition  $[0, \delta_0] = \cup_{n \geq 1} [\delta_0 2^{-n}, \delta_0 2^{-(n-1)}]$ , we consider the complementary sets in  $[\frac{\delta_1}{2}, \delta_1]$

$$\begin{aligned} (\mathcal{G}_{\gamma_1,\kappa_0}^{(N')}(u_2))^c \setminus (\mathcal{G}_{\gamma_1,\kappa_0}^{(N)}(u_1))^c &= (\mathcal{G}_{\gamma_1,\kappa_0}^{(N')}(u_2))^c \cap \mathcal{G}_{\gamma_1,\kappa_0}^{(N)}(u_1) \\ &\subset [\cup_{r \leq N} (\mathcal{G}_r^c(u_2) \cap \mathcal{G}_r(u_1) \cap \mathcal{G})] \cup [\cup_{r > N} \mathcal{G}_r^c(u_2) \cap \mathcal{G}]. \end{aligned}$$

If  $r \leq N_{\varepsilon_0}$ , then  $\mathcal{G}_r^c(u_2) \cap \mathcal{G} = \emptyset$ . So it is sufficient to prove that, if  $\|u_1 - u_2\|_{\bar{\sigma}} \leq N^{-e}$ ,  $e \geq d + n + 3$ , then

$$\Omega := \sum_{N_{\varepsilon} < r \leq N} |\mathcal{G}_r^c(u_2) \cap \mathcal{G}_r(u_1)| + \sum_{r > \max\{N, N_{\varepsilon}\}} |\mathcal{G}_r^c(u_2)| \leq C'\gamma_1\delta_1 N^{-1}.$$

Note that  $\|(L_a^{(r)})^{-1}\|_0$  is the inverse of the eigenvalue of smallest modulus and

$$\|L_a^{(r)}(u_2) - L_a^{(r)}(u_1)\|_0 = O(\varepsilon\|u_2 - u_1\|_{s_0}) = O(\varepsilon N^{-e}).$$

The sufficient and necessary condition of an eigenvalues of  $L_a^{(r)}(u_2)$  in  $[-4\gamma_1 r^{-\tau} - C\varepsilon N^{-e}, 4\gamma_1 r^{-\tau} + C\varepsilon N^{-e}]$  is that there exists an eigenvalues of  $L_a^{(r)}(u_1)$  in  $[-4\gamma_1 r^{-\tau}, 4\gamma_1 r^{-\tau}]$ . Thus, it leads to

$$\begin{aligned} \mathcal{G}_r^c(u_2) \cap \mathcal{G}_r(u_1) &\subset \{\delta \in [\frac{\delta_1}{2}, \delta_1] \mid \exists \text{ at least an eigenvalue of } L_a^{(r)}(\delta, u_1) \\ &\quad \text{with modulus in } [4\gamma_1 r^{-\tau}, 4\gamma_1 r^{-\tau} + C\varepsilon N^{-e}]\}. \end{aligned}$$

Next we claim that if  $\varepsilon$  is small enough and  $I$  is a compact interval in  $[-\gamma_1, \gamma_1]$  of length  $|I|$ , then

$$\begin{aligned} |\{\delta \in [\frac{\delta_1}{2}, \delta] \text{ s.t. at least } \exists \text{ an eigenvalue of } L^{(r)}(\delta, u_1) \text{ belongs to } I\}| \\ \leq C r^{d+n+1} \delta_1^{-(2\rho-2)} |I|. \end{aligned} \quad (71)$$

Due to the  $C^1$  map  $\delta \mapsto L^{(r)}(\delta, u_1)$  and the selfadjoint property of  $L^{(r)}(\delta, u_1)$ , we have the corresponding eigenvalue function  $\lambda_k(\delta, u_1)$  with  $1 \leq k \leq r$ . Denote the eigenspace of  $L^{(r)}(\delta, u_1)$  by  $E_{\delta,k}$  associated to  $\lambda_k(\delta, u_1)$ , then by  $\|\partial_{\delta} b\|_s = \|(\partial_u^2 f)(x, u)\|_s \leq C\gamma_1^{-1}$  and  $\|\nabla^{\rho} h\|_0^2 \geq \|h\|_0^2$ , for sufficient small  $0 < \varepsilon \leq \varepsilon_0(\gamma_1)$ , we have

$$\begin{aligned} (\partial_{\delta} \lambda_k(\delta, u_1)) &\leq \max_{h \in E_{\delta,k}, \|h\|_0=1} \left( (\partial_{\delta} L^{(r)})(\delta, u_1) h, h \right)_0 \\ &\leq \max_{h \in E_{\delta,k}, \|h\|_0=1} \left( (2\rho - 1) \delta^{2\rho-2} (\Delta^{\rho} h, h)_0 + O(\varepsilon \gamma_1^{-1}) \right) \\ &\leq \max_{h \in E_{\delta,k}, \|h\|_0=1} \left( -(2\rho - 1) \delta^{2\rho-2} \|\nabla^{\rho} h\|_0^2 + O(\varepsilon \gamma_1^{-1}) \right) \\ &\leq \max_{h \in E_{\delta,k}, \|h\|_0=1} \left( -(2\rho - 1) \delta^{2\rho-2} \|h\|_0^2 + O(\varepsilon \gamma_1^{-1}) \right) \\ &\leq -(2\rho - 1) \delta^{2\rho-2} + O(\varepsilon \gamma_1^{-1}) \leq -2(\rho - 1) \delta_1^{2\rho-2}. \end{aligned}$$

Hence we have  $|\lambda_k^{-1}(I, u_1) \cap [\frac{\delta_1}{2}, \delta_1]| \leq C|I|\delta_1^{-(2\rho-2)}$ . The claim holds.

Thus, we obtain

$$|\mathcal{G}_r^c(u_2) \cap \mathcal{G}_r(u_1)| \leq C\varepsilon r^{d+n+1}\delta_1^{-(2\rho-2)}N^{-e} \leq C\delta_1 N^{-e}r^{d+n+1}.$$

Furthermore, by (71), we have  $|\mathcal{G}_r^c(u_2)| \leq C\gamma_1 r^{d+n-\tau+1}\delta_1^{-(2\rho-2)}$ . Therefore, we obtain

$$\begin{aligned} \Omega &= \sum_{N_\varepsilon < r \leq N} |\mathcal{G}_r^c(u_2) \cap \mathcal{G}_r(u_1)| + \sum_{r > \max\{N, N_\varepsilon\}} |\mathcal{G}_r^c(u_2)| \\ &\leq C\delta_1 \left( \sum_{r \leq N} r^{d+n+1} \right) N^{-e} + C\gamma_1 \delta_1^{-(2\rho-2)} \sum_{r > \max\{N, N_\varepsilon\}} r^{d+n-\tau+1} \\ &\leq C' \left( \delta_1 N^{d+n-e+2} + \gamma_1 \delta_1^{-(2\rho-2)} (\max\{N, N_\varepsilon\})^{d+n-\tau+2} \right) \\ &\leq C'' \gamma_1 \delta_1 N^{-1}, \end{aligned}$$

where  $C$ ,  $C'$  and  $C''$  denote constants. This completes the proof.

**Proof of Lemma 18.** The key step of Lemma 18 is the following Theorem 8. To prove Theorem 8, we only need to give the proof of Lemma 25, the remainder of the proof is the same as [2], so we omit it.

Define the bilinear symmetric form  $\phi_\varepsilon : \mathbf{R}^{r+n} \times \mathbf{R}^{r+n} \rightarrow \mathbf{R}$  by

$$\phi_\varepsilon(x, x') := J \cdot J' - \varepsilon a J^* \cdot J^{*'}, \quad \forall J \in \mathbf{R}^{r+n},$$

where  $x = (J, J^*)$ ,  $x' = (J', J^{*'}) \in \mathbf{R}^{r+n} \times \mathbf{R}^{r+n}$  and choose  $J^* \in \mathbf{R}^{r+n}$  such that the corresponding quadratic form

$$\Phi_\varepsilon(x) = \phi_\varepsilon(x, x) = |J|^2 - \varepsilon a |J|^{2e}.$$

Denote  $x = j' + \vec{\rho}' = (J, J^*)$ , where  $\vec{\rho}' = (\rho, 0, 0, 0)$ ,  $\forall j' = (j_1, j_2, j_1^*, j_2^*) \in \Lambda^+ \times \mathbf{Z}^n \times \Lambda^+ \times \mathbf{Z}^n$  and  $x \in \Lambda^{++} \times \mathbf{Z}^n \times \Lambda^+ \times \mathbf{Z}^n$  since  $j_1 \in \Lambda^+$  and  $\Lambda^{++} = \rho + \Lambda^+$ . Note that  $\Phi_\varepsilon(j' + \vec{\rho}') = D_j + |\rho|^2$ , where  $D_j$  are the small divisors. We say a vector  $x = (j' + \rho, j^{*'}) \in \Lambda^{++} \times \mathbf{Z}^n \times \Lambda^{++} \times \mathbf{Z}^n$  is "weak singular" if  $|\Phi_\varepsilon(x)| \leq C$  for some constant  $C$  fixed once and for all.

**Definition 3.** A sequence  $x_0, x_1, \dots, x_K \in \Lambda^{++} \times \mathbf{Z}^n \times \Lambda^{++} \times \mathbf{Z}^n$  of distinct, weakly singular vectors satisfying, for some  $B \geq 2$ ,  $|x_{k+1} - x_k| \leq B$ ,  $\forall k = 0, 1, \dots, K-1$ , is called a  $B$ -chain of length  $K$ .

**Theorem 10** Assume that  $\varepsilon$  satisfies (5). Then any  $B$ -chain has length  $K \leq B^C \gamma^{-p}$  for some  $C := C(G) > 0$  and  $p := p(G) > 0$ .

Using Lemma 2, we can easily prove the following result. It can be found in [2].

**Lemma 25** Let  $\mathcal{M} = (G \times \mathbf{T}^n)/N$ . The matrices  $R$  and  $S$  have coefficient in  $D^{-1}\mathbf{Z}$  for some  $D \in \mathbf{N}$ .

Given lattice vectors  $f_i \in \Lambda \times \mathbf{Z}^n \times \Lambda \times \mathbf{Z}^n$ ,  $i = 1, \dots, n$ ,  $1 \leq m \leq 2r + 2n$ , linearly dependent on  $\mathbf{R}$ , we consider the subspace  $F := \text{Span}_{\mathbf{R}}\{f_1, \dots, f_m\}$  of  $\mathbf{R}^{r+n} \times \mathbf{R}^{r+n}$  and the restriction  $\phi_\varepsilon|_F$  of the bilinear form  $\phi_\varepsilon$  to  $F$ , which is represented by the symmetric matrix  $A_\varepsilon := \{\phi_\varepsilon(f_i, f_{i'})\}_{i,i'=1}^m$ . Denote  $\varphi(x, x') := J \cdot J'$  and  $\varphi^*(x, x') := J^* \cdot J^{*'} the symmetric bilinear forms. Then we rewrite$

$$\Phi_\varepsilon = \varphi - \varepsilon a \varphi^*, \quad A_\varepsilon = R - \varepsilon a S,$$

where

$$R := \{\varphi(f_i, f_{i'})\}_{i,i'=1}^m = (a_{ii'})_{i,i'=1}^m, \quad S := \{\varphi^*(f_i, f_{i'})\}_{i,i'=1}^m = (b_{ii'})_{i,i'=1}^m$$

are the matrices that represent, respectively,  $a_{ii'}$  and  $b_{ii'}$ ,  $i, i' = 1, \dots, m$  denote the element, respectively, of  $R$  and  $S$ ,  $\varphi|_F$  and  $\varphi^*|_F$  in the basis  $\{f_1, \dots, f_m\}$ .

Since the matrix  $S$  is not at most rank 1, the proof of following result is some what different from the proof of Lemma A.3 in [2]. But the main idea is the same.

**Lemma 26** *Assume that  $a$  satisfies (5). Then  $A_\varepsilon$  satisfies*

$$\|A_\varepsilon^{-1}\| \leq \frac{c(m, D)}{\gamma} \left( \max_{i=1, \dots, m} |f_i| \right)^{5m-2},$$

where  $c(m, D)$  is a constant depending on  $m$  and  $D$ .

*Proof* Direct calculation shows

$$\begin{aligned} \det A_\varepsilon &= \det(R - \varepsilon a S) \\ &= \sum_{j_1, j_2, \dots, j_m} (-1)^{\tau(j_1, j_2, \dots, j_m)} (a_{1j_1} - \varepsilon a b_{1j_1}) (a_{2j_2} - \varepsilon a b_{2j_2}) \cdots (a_{mj_m} - \varepsilon a b_{mj_m}) \\ &= \det R + (-1)^n \varepsilon^m a^m \det S + P'(\varepsilon), \end{aligned} \quad (72)$$

where  $P'(\varepsilon)$  is a polynomial on  $\varepsilon$  of degree  $m-1$  with integer coefficients (by Lemma 23), and  $\tau(j_1, j_2, \dots, j_m)$  is the rank of  $j_1, \dots, j_m$ .

Note that  $\det R, \det S, P'(1) \in D^{-m} \mathbf{Z}$ . By Lemma 23,  $D^m \det A_\varepsilon = P(\varepsilon)$  is a polynomial on  $\varepsilon$  of degree  $n$  with integer coefficients. It follows from  $P(-a) = D^m \det(R + a^2 S)$  that  $P(\cdot) \neq 0$ . By (72), if  $\det R = 0$ , then  $|\det A_\varepsilon| \geq \varepsilon^n a^n D^{-m}$ . If  $\det R \neq 0$ , then by (22), we have

$$|\det A_\varepsilon| \geq \gamma D^{\frac{5m}{2}} |\det R|^{-\frac{3}{2}}. \quad (73)$$

We can write  $R + a^2 S = \xi^T \xi$  with  $\xi = (f_1, \dots, f_m)$ . Thus we have

$$0 \leq \det R \leq \det(R + S) = (\det \xi)^2 \leq |f_1|^2 \cdots |f_m|^2 \leq M^{2m}, \quad (74)$$

where  $M := \max_{i=1, \dots, m} |f_i|$ .

By (73)-(74), we derive

$$|\det A_\varepsilon| \geq \gamma D^{\frac{5m}{2}} |\det R|^{-3m}. \quad (75)$$

Note that (22),  $a \geq \gamma$  and (75) hold. Using the Cramer rule and (75), we can obtain the main result. This completes the proof.

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